On the Role of Regularity in Mathematical Control Theory

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March 18, 2016

Introduction

- Regularity of mappings plays an essential role in mathematical control theory.
- Few examples: theory of stabilizability, theory of accessibility.
- Real analytic vs. smooth
- Mathematics and Physics

Time-varying vector fields and their flows

- Time-varying vector fields with **measurable** dependence on time arise naturally in control theory.
- An operator approach for studying time-varying vector fields and their flows.
- The $\mathbb{R}$-algebra $C^\nu(M)$ and a suitable locally convex topology on $C^\nu(M)$ plays an essential role in this framework.

C^ν-vector fields

There is a one-to-one correspondence between C^ν-vector fields on M and derivations of the $\mathbb{R}$-algebra $C^\nu(M)$.

C^ν-maps

There is a one-to-one correspondence between C^ν-maps from M to M and unital-algebra homomorphisms of the $\mathbb{R}$-algebra $C^\nu(M)$.

Topology on space of C^ν-sections

- Let (E, M, π) be a C^ν-vector bundle. The C^∞-topology on $\Gamma^\infty(E)$ is classical.

C^∞-topology

The locally convex C^∞-topology on $\Gamma^\infty(E)$ is defined using a family of seminorms

$$p^\infty_{K,m} = \sup\left\{ \|D^{(r)}X(x)\| \;\middle|\; x \in K, |(r)| \leq m \right\}.$$

C^{hol}-topology

The C^{hol}-topology on the space $\Gamma^{\mathrm{hol}}(E)$ is defined as the compact-open topology, i.e., the topology generated by the family of seminorms

$$p^{\mathrm{hol}}_K = \sup\left\{ \|X(x)\| \mid x \in K \right\}.$$

Topology on space of C^ν-sections

- The C^ω-topology on $\Gamma^\infty(E)$ is defined using the fact that a real analytic section can be considered as a **germ** of a holomorphic section on a suitable domain.

C^ω-topology

The C^ω-topology on $\Gamma^\omega(E)$ is defined as the following inductive limit topology (i.e., the finest locally convex topology) such that, for every complex vector bundle $\overline{U}$ containing E, the restriction map

$$r_{\overline{U}} : \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{U}) \to \Gamma^\omega(E)$$

is continuous.

Equivalent characterization of C^ω-topology

Each of the following define the C^ω-topology on $\Gamma^\omega(E)$.

1. The **inductive limit topology** defined as the finest locally convex topology on $\Gamma^\omega(E)$ such that, for every complex vector bundle $\overline{U}$ containing E, the restriction map

$$r_{\overline{U}} : \Gamma^{\mathrm{hol},\mathbb{R}}(\overline{U}) \to \Gamma^\omega(E)$$

   is continuous.

2. The **projective limit topology** defined as the coarsest locally convex topology on $\Gamma^\omega(E)$ such that, for every compact set $K \subseteq E$, the inclusion map

$$i_M : \Gamma^\omega(E) \to \mathscr{G}_K^{\mathrm{hol},\mathbb{R}}$$

   is continuous.

3. The topology generated by the family of seminorms $\{p^\omega_{K,\mathbf{a}}\}$, where $K \subseteq M$ is a compact set and $\mathbf{a} \in \mathbf{c}_0^\downarrow(\mathbb{Z}_{\geq 0}; \mathbb{R}_{>0})$.

$$p^\omega_{K,\mathbf{a}}(X) = \sup\left\{ \frac{a_0 a_1 \ldots, a_{|r|}}{r!} \|D^{(r)}X(x)\| \;\middle|\; x \in K, |r| \in \mathbb{Z}_{\geq 0} \right\}$$

Linear mappings

Space of linear continuous mappings

The space of linear continuous mappings from $C^{\nu}(M)$ to $C^{\nu}(M)$ is denoted by $\mathrm{L}(C^{\nu}(M); C^{\nu}(M))$.

- The space $\mathrm{L}(C^{\nu}(M); C^{\nu}(M))$ contains both C^{ν}-vector fields on M and C^{ν}-maps from M to M.
- The topology of pointwise convergence on $\mathrm{L}(C^{\nu}(M); C^{\nu}(M))$ is denoted by C^{ν}-topology.

Time-varying vector fields

- A complete time-varying vector fields and their flows are curves on $\mathrm{L}(C^\nu(M); C^\nu(M))$.

Time-varying vector field

For every time-varying vector field $X : \mathbb{T} \times M \to TM$, we define a curve $\widehat{X} : \mathbb{T} \to \mathrm{L}(C^\nu(M); C^\nu(M))$ as

$$\widehat{X}(t)(f) = X_t f.$$

- There is a one-to-one correspondence between time-varying C^ν-vector fields on M and curves on $\mathrm{L}(C^\nu(M); C^\nu(M))$ with the property that

$$\widehat{X}(t)(fg) = \widehat{X}(t)(f)g + f\widehat{X}(t)(g)$$

Flows of time-varying vector fields

Time-varying C^ν-maps

For every time-varying mapping $\phi : \mathbb{T} \times U \to M$, we define a curve $\widehat{\phi} : \mathbb{T} \to \mathrm{L}(C^\nu(M); C^\nu(U))$ as

$$\widehat{\phi}(t)(f) = f \circ \phi_t.$$

- There is a one-to-one correspondence between time-varying C^ν-maps on M and curves on $\mathrm{L}(C^\nu(M); C^\nu(M))$ with the property that

$$\widehat{\phi}(t)(fg) = \widehat{\phi}(t)(f)\widehat{\phi}(t)(g)$$

Bochner integrability

- $\mathrm{L}(C^\nu(M); C^\nu(M))$ is a locally convex space. In order to study curves on this space, we need to define different notions on locally convex spaces

Characterization of Bochner integrable curves

Let V be a complete and separable locally convex space and $\{p_\alpha\}_{\alpha\in\Lambda}$ be a family of generating seminorms on V.
A curve $\gamma : \mathbb{T} \to V$ is Bochner integrable if and only if, for every $\alpha \in \Lambda$, the curve

$$t \mapsto \int_{\mathbb{T}} p_\alpha(\gamma(\tau))d\tau,$$

is integrable.

- The space of Bochner integrable curves from the interval $\mathbb{T}$ to the locally convex space V is denoted by $\mathrm{L}^1(\mathbb{T}; V)$.

Absolute continuity

Absolute continuity

Let V be a locally convex space and $\{p_\alpha\}_{\alpha \in \Lambda}$ be a family of generating seminorms on V. Then a curve $\gamma : \mathbb{T} \to V$ is absolutely continuous if there exists a Bochner integrable curve $\eta : \mathbb{T} \to V$ such that

$$\gamma(t) = \gamma(t_0) + \int_{t_0}^{t} \eta(\tau) d\tau.$$

- The space of absoluelty continuous curves from the interval $\mathbb{T}$ to the locally convex space V is denoted by $\mathrm{AC}(\mathbb{T}; V)$.

Holomorphic extension of real analytic time-varying vector fields

- It is well-known that every real analytic **time-invariant** vector field can be homomorphically extended to a holomorphic vector field on a suitable domain.
- Does the same result hold for **time-varying** vector fields? Not generally.

Example

The time-varying vector field $X : \mathbb{R} \times \mathbb{R} \to T\mathbb{R}$ defined by

$$X(t,x) = \begin{cases} \frac{t^2}{t^2+x^2}\frac{\partial}{\partial x} & x \neq 0 \text{ or } t \neq 0, \\ 0 & x, t = 0. \end{cases}$$

is locally integrable in t when x is fixed and real analytic in x when t is fixed. However, there does not exist any interval of time $T \subseteq \mathbb{R}$ containing $t = 0$ and any neighbourhood $\overline{U} \subseteq \mathbb{C}$ of $x = 0$ such that X can be extended to a holomorphic vector field on $T \times \overline{U}$

Holomorphic extension of real analytic time-varying vector fields

- We need some extra conditions to ensure existence of holomorphic extension. Bochner integrability do this for us.

Global extension of real analytic vector fields

Let $X : \mathbb{T} \times M \to TM$ be a locally Bochner integrable time-varying real analytic vector field. Then there exists a complex manifold $\overline{U}$ containing M and a locally Bochner integrable time-varying holomorphic vector field $\overline{X} : \mathbb{T} \times \overline{U} \to T\overline{U}$ such that

$$\overline{X}(t,x) = X(t,x), \qquad \forall t \in \mathbb{T},\ \forall x \in M.$$

- Idea of proof: Using the inductive limit characterization of $\Gamma^{\omega}(TM)$.

Holomorphic extension of real analytic time-varying vector fields

- What happen if we have a "couple" of time-varying real-analytic vector fields and we want to ensure that there exists a "single" domain on which all these real analytic vector fields can extend to holomorphic vector fields?

Local extension of real analytic vector fields

Let B be a bounded set in $\mathrm{L}^1(\mathbb{T};\Gamma^\omega(TM))$. Then, for every compact set $K \subseteq U$, there exits a complex manifold $\overline{U}_K$ containing K such that, for every $X \in B$, there exists $\overline{X} : \mathbb{T} \times \overline{U}_K \to T\overline{U}_K$ such that

$$\overline{X}(t,x) = X(t,x), \qquad \forall t \in \mathbb{T},\ \forall x \in K.$$

- Idea of the proof: Using the projective limit characterization of $\Gamma^\omega(TM)$.

Flows of time-varying vector fields

- One of the main reasons for developing this operator framework is to study "flows" of time-varying vector fields.
- From nonlinear differential equations to linear differential equations.

Nonlinear vs Linear

Nonlinear DE	Linear DE
$\frac{dx}{dt} = X(t,x),$ $x(t_0) = x_0.$	$\frac{d}{dt}\phi(t) = \phi(t)\circ X(t),$ $\phi(t_0) = \mathrm{id}.$

- While this approach translate our nonlinear differential equation into a linear differential equation, we pay a cost: The resulting linear differential is on an infinite dimensional vector space $\mathrm{L}(C^\nu(M); C^\nu(M))$.

Flows of time-varying vector fields

- Theory of linear differential equations on locally convex spaces is completely different from the theory of linear differential equations on Banach spaces.
- Picard iterations for linear differential equations.

Iterative Solution

We define $\phi_0(t) \in \mathrm{L}^1(C^\omega(M); C^\omega(U))$ as

$$\phi_0(t)(f) = f\,|_U, \qquad \forall t \in [0, T].$$

and we define $\phi_N(t)$ inductively as

$$\phi_N(t) = \phi_0(t) + \int_{t_0}^{t} \phi_{N-1}(\tau) \circ X(\tau) d\tau.$$

Flows of time-varying vector fields

- One can show that for "locally Bochner integrable" time-varying real analytic vector fields, the sequence of Picard iterations converges uniformly on small-enough interval $[0, T]$ in C^ω-topology to the flow of X.

Theorem

Let $\widehat{X} : \mathbb{T} \to \Gamma^\nu(TM)$ be a locally Bochner integrable time-varying C^ν-vector field on M. Then, for every $x_0 \in M$ there exist a neighbourhood U of x_0 and a locally absolutely continuous curve $\widehat{\phi} : \mathbb{T} \to \mathrm{L}(C^\nu(M); C^\nu(U))$ such that

$$\begin{aligned} \frac{d\widehat{\phi}(t)}{dt} &= \widehat{\phi}(t) \circ \widehat{X}(t), \qquad \text{a.e. } t \in \mathbb{T}, \\ \widehat{\phi}(t_0) &= \mathrm{id}. \end{aligned}$$

Moreover, for almost every $t \in \mathbb{T}$, we have

$$\widehat{\phi}(t)(fg) = \widehat{\phi}(t)(f)\widehat{\phi}(t)(g).$$

Exponential map

- There does not exist a "single" neighbourhood U such that, for every $X \in \mathrm{L}^1(\mathbb{T}; \Gamma^\nu(TM))$, the flow of X (i.e., ϕ^X) is defined on U.
- Therefore, in order to define a relation between time-varying vector fields and their flows, it should be defined on germs of vector fields and germs of flows.

Exponential map

The exponential map defined in this manner is sequentially continuous.

- In smooth case, this result is classical. For the real analytic case it is proved using the approximations for sequence of Picard iterations.

Summary

- A coherent framework where all the regularity classes (in particular the important class of real analytic vector fields) can be treated essentially the same.
- Global and local extension of "locally Bochner integrable" time-varying real analytic vector fields.
- Convergence of the sequence of Picard iteration in C^ω-topology.
- Defining exponential map using the germ of vector fields and their flows and showing that it is sequentially continuous.

Introduction to tautological control systems

- In geometric control theory, we define a control system using a family of parametrized C^{ν}-vector fields $\{F_u\}_{u \in \mathcal{U}}$, where the parameter is called **control**.
- Most fundamental properties of control systems (for example controllability, accessibility, ...) depends on the trajectories of the system.
- Most methodologies in control literature for studying fundamental properties of control systems are not **parameter-invariant**.

Tautological control systems

Example

Consider these two control systems

$$\begin{aligned} \dot{x}_1 &= u_1, \\ \dot{x}_2 &= u_2, \\ \dot{x}_3 &= x_1^2 - x_2^2. \end{aligned} \qquad \begin{aligned} \dot{x}_1 &= \tfrac{u_1+u_2}{2}, \\ \dot{x}_2 &= \tfrac{u_1-u_2}{2}, \\ \dot{x}_3 &= x_1^2 - x_2^2. \end{aligned}$$

By considering the control set $\mathcal{U} = \mathbb{R}^2$, these two systems have the same trajectories.
Sussmann's sufficient condition can show that the left system is small-time locally controllable but it is indecisive about the right system.

- Instead of working with **parametrized** family of vector fields, we define a control system as a **presheaf** of vector fields

Tautological control systems and control

C^ν-tautological control system

A C^ν-tautological control system is a pair $(M, \mathscr{F})$, where

1. M is a C^ν-manifold called **state manifold**, and
2. $\mathscr{F}$ is a presheaf of C^ν-vector fields.

- The vector fields in $\mathscr{F}$ may be locally defined. Thus tautological control systems are "generalization" of classical control systems.
- A C^ν-tautological control system is **globally generated**, if every vector field in $\mathscr{F}$ is globally defined.
- What is the relationship between notions of tautological control systems and C^ν-control systems?

Tautological control systems and C^ν-control systems

- Given a C^ν-control system $\Sigma = (M, F, \mathcal{U})$, we define the presheaf $\mathscr{F}_\Sigma$ as

$$\mathscr{F}_\Sigma(U) = \{F^u \,|_U \mid u \in \mathcal{U}\}\,.$$

Theorem

Let $\Sigma = (M, F, \mathcal{U})$ be a C^ν-control system, then the pair $\Theta = (M, \mathscr{F}_\Sigma)$ is a C^ν-tautological control system associated to Σ

- The above result shows that, one can associate a C^ν-tautological control system $(M, \mathscr{F}_\Sigma)$.

Tautological control systems and C^ν-control systems

- Is this correspondence one-to-one? The answer is no generally!
- However, if the tautological control system is globally generated, the answer is "Yes".
- Let $\Theta = (M, \mathscr{F})$ be a globally generated C^ν-tautological control system. Then we define the family of parametrized vector field $F_\Theta : \mathscr{F} \times M \to TM$ as

$$F_\Theta(X, x) = X(x), \qquad \forall X \in \mathscr{F}.$$

Theorem

Let $\Theta = (M, \mathscr{F})$ be a globally generated C^ν-tautological control system. Then $\Sigma = (M.F_\Theta, \mathscr{F})$ is a C^ν-control system.

Trajectories of tautological control systems

- For classical control systems, a trajectory of the system is defined as the trajectory of a time-varying vector field obtained by "plugging in" an admissible control.
- For tautological control systems, we define "open-loop system" and then trajectories of the systems are defined as the trajectories of the open-loop system.
- A "naive" way of defining open-loop families is to define it as the family of all locally Bochner integrable time-varying C^ν-vector field $X : \mathbb{T} \times U \to TU$ such that

$$\widehat{X}(t) \in \mathscr{F}(U), \qquad \forall t \in \mathbb{T}.$$

Trajectories of tautological control systems

Example

Consider the C^ω-tautological control system $\Sigma = (\mathbb{R}^2, \mathscr{F})$ as

$$\mathscr{F}(U) = \begin{cases} \{\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\} & (0,0) \notin U, \\ \{\frac{\partial}{\partial x}\} & (0,0) \in U. \end{cases}$$

The curve $\gamma : [0,1] \to \mathbb{R}^2$ defined as

$$\gamma(t) = \begin{cases} (0, t) & 0 \leq t \leq \frac{1}{2}, \\ (t - \frac{1}{2}, \frac{1}{2}) & \frac{1}{2} \leq t \leq 1. \end{cases}$$

is concatenation of trajectories of Σ, but it is not a trajectory of Σ.

- This problem can be resolved using the "germs" of vector fields.

Trajectories of tautological control systems

- We define the sheaf $\mathrm{LI}\mathscr{F}^\nu$ as

$$\mathrm{LI}\mathscr{F}^\nu(\mathbb{T} \times U) = \mathrm{L}^1(\mathbb{T}; \mathscr{F}(U))$$

- Note that

$$\mathrm{L}^1(\mathbb{T}; \mathscr{F}(U)) = \{X \in \mathrm{L}^1(\mathbb{T}; \Gamma^\nu(U)) \mid X(t) \in \mathscr{F}(U), \text{ a.e. } t \in \mathbb{T}\}.$$

Etalé open-loop systems and etalé trajectories

An **etalé open-loop system** for the C^ν-tautological control system $\Sigma = (M, \mathscr{F})$ an element in $\mathrm{Sh}(\mathrm{LI}\mathscr{F}^\nu)(W)$, for some open set $W \subseteq M$. An **etalé trajectory** of Σ is a locally absolutely curve $\gamma : \mathbb{T} \to M$ such that there exists an open-loop system X for Σ such that

$$\frac{d\gamma(t)}{dt} = \mathrm{ev}_{(t,\gamma(t))}\left(X(t, \gamma(t))\right), \quad \text{a.e } t \in \mathbb{T}.$$

Summary

- Develop a parameter-invariant framework for studying control systems.
- Using the notion of sheaf of vector fields. we define tautological control systems.
- Study how the new notion of control systems is connected to the classical one.
- We define the trajectories for tautological control systems in a suitable manner.

Orbit theorem in classical control theory

- Chow-Rashevskii theorem.
- One can generalize Chow-Rashevskii theorem for distributions that has one of the following properties
  1. The distribution locally has a constant rank.
  2. The distribution is real analytic.
  3. The C^ν-module generated by the Lie brackets of vector fields of the distribution is "locally finitely generated".
- Sussmann and Stefan independently proved a singular version of the Chow-Rashevskii called the "orbit theorem"

Tautological orbit theorem

Let $\Sigma = (M, \mathscr{F})$ be a C^ν-tautological control system.

- The orbit of Σ passing through $x \in M$ is

$$\mathrm{Orb}_\Sigma(x) = \left\{ \phi^{X_1}_{t_1} \circ \phi^{X_2}_{t_2} \circ \ldots \circ \phi^{X_m}_{t_m}(x) \;\middle|\; X_i \in \mathscr{F}(U_i), m \in \mathbb{Z}_{\geq 0} \right\}$$

- We define a presheaf of modules $\overline{\mathscr{F}}$ which assigns to every open set $U \subseteq M$ the following $C^\nu(U)$-module:

$$\overline{\mathscr{F}}(U) = \mathrm{span}_{C^\nu(U)}\{\eta^* X \mid \exists V \text{ an open subset of } M \text{ s.t. } X \in \mathscr{F}(V),\ \eta : U \to V \in \Gamma(\mathscr{P}_{\mathscr{F}})\},$$

Tautological orbit theorem

For every $x \in M$, the orbit $\mathrm{Orb}_\Sigma(x)$ has a unique structure as an immersed submanifold of M. Moreover, for every $y \in \mathrm{Orb}_\Sigma(x)$, we have

$$T_y \mathrm{Orb}_\Sigma(x) = \overline{\mathscr{F}}(y).$$

Tautological orbit theorem

- The presheaf $\overline{\mathscr{F}}$ involve the flows of the system. The computation of elements in $\overline{\mathscr{F}}$ is complicated.
- By imposing some extra conditions, can we characterize $\overline{\mathscr{F}}(x)$ using something simpler to compute, for instance $\mathrm{Lie}(\mathscr{F})(x)$?
- Note that the presheaf $\mathrm{Lie}(\mathscr{F})(x)$ is defined as
- If $\mathscr{F}$ is real analytic, then is it true that $\overline{\mathscr{F}}(x) = \mathrm{Lie}(\mathscr{F})(x)$, for every $x \in M$?
- If, for every small-enough U, the module $\mathrm{Lie}(\mathscr{F})(U)$ is locally finitely generated, then is it true that $\overline{\mathscr{F}}(x) = \mathrm{Lie}(\mathscr{F})(x)$, for every $x \in M$?
- Generally, the answer to the above questions is negative.

Tautological orbit theorem

Example

Let $\mathscr{F} = \{X_1, X_2\}$, where $X_1 : \mathbb{R}^2 \to T\mathbb{R}^2$ is defined as

$$X_1(x, y) = \frac{\partial}{\partial x},$$

and $X_2 : \mathbb{R}^{>0} \times \mathbb{R} \to T\mathbb{R}^2$ is defined as

$$X_2(x, y) = \frac{1}{x}\frac{\partial}{\partial y}.$$

1. the vector fields in $\mathscr{F}$ are not globally defined,
2. for every x, there exists a neighbourhood $U \subseteq \mathbb{R}^2$ of x such that $\mathrm{Lie}(\mathscr{F})(U)$ is a locally finitely generated $C^\omega(U)$-module, and
3. $\mathrm{Lie}(\mathscr{F})(0,0) \neq \overline{\mathscr{F}}(0,0)$.

Tautological orbit theorem

- If we replace the condition that the "module" $\mathrm{Lie}(\mathscr{F})$ is locally finitely generated with the condition that the "presheaf" is locally finitely generated, then we have

Theorem

Let $\mathscr{F}$ be a presheaf of C^ν-vector fields such that $\mathrm{Lie}(\mathscr{F})$ is a locally finitely generated presheaf. Then we have

$$\overline{\mathscr{F}}(x) = \mathrm{Lie}(\mathscr{F})(x), \qquad \forall x \in M.$$

Real analytic case

- Let $\{s_\alpha\}_{\alpha\in\Lambda}$ be a family of real analytic vector fields on M. Then the presheaf of C^ω-modules generated by $\{s_\alpha\}_{\alpha\in\Lambda}$ is locally finitely generated.

Theorem

Let $\mathscr{F}$ be a "globally generated" presheaf of C^ω-vector fields. Then we have

$$\overline{\mathscr{F}}(x) = \mathrm{Lie}(\mathscr{F})(x), \qquad \forall x \in M.$$

Summary

- Generalize the orbit theorem for tautological control systems.
- Orbits are C^ν-immersed submanifolds. Their tangent space at each point is described by the presheaf of modules $\overline{\mathscr{F}}$.
- The sheaf structure of $\mathrm{Lie}(\mathscr{F})$ plays a crucial role in characterizing the tangent space to the orbits.
- If the "presheaf" $\mathrm{Lie}(\mathscr{F})$ is locally finitely generated, then $\mathrm{Lie}(\mathscr{F})(x) = \overline{\mathscr{F}}(x)$