Synchronization and Multistability in Oscillator Networks and Power Grids

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Introduction: Coupled oscillators Network

Pendulum clocks: "an odd kind of sympathy"
[Christiaan Huygens, Horologium Oscillatorium, 1673]

Models for coupled oscillators:
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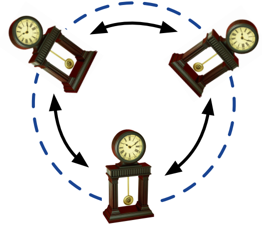

Kuramoto model

1. **n-oscillators** with phases θ_i,
2. with natural frequencies $\omega_i \in \mathbb{R}$,
3. **coupling** with strength $a_{ij} = a_{ji}$.

$$\dot{\theta}_i = \omega_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j).$$

Application: Active power dynamics

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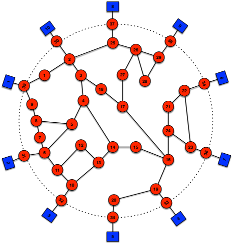
    2. **quasi-sync:** voltage and phase V_i, θ_i
       active power p_i
3. simplifying assumptions:
    1. lossless and inductive lines with admittances Y_{ij}
    2. decoupling of phase and voltage dynamics



New England IEEE 39-bus

Application: Active power dynamics

Structure-Preserving Model [A. Bergen & D. Hill, 1981]:

Active power dynamics

Generators: $$M_i\ddot{\theta}_i + D_i\dot{\theta}_i = p_i - \sum_j a_{ij}\sin(\theta_i - \theta_j)$$

Inverters: $$\Lambda_i\dot{\theta}_i = p_i - \sum_j a_{ij}\sin(\theta_i - \theta_j)$$

Loads: $$\tau_i\dot{\theta}_i = p_i - \sum_j a_{ij}\sin(\theta_i - \theta_j)$$

where

Active power capacity of line (i,j): $a_{ij} = |Y_{ij}|V_iV_j$

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Frequency synchronization = Operating point = Equilibriums

$$p_i = \sum_{j=1}^{n} a_{ij}\sin(\theta_i - \theta_j)$$

Synchronization problem

Frequency synchronization = Operating point = Equilibriums

$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

Key questions

Given the network and the power profile:

Q1: does there exist a **stable equilibrium point**?

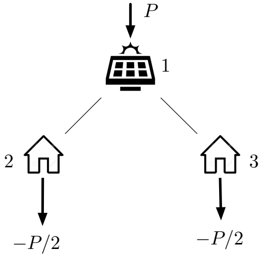
Q2: is this stable equilibrium point **unique**?

Q3: how to measure the **robustness** of the synchronization?

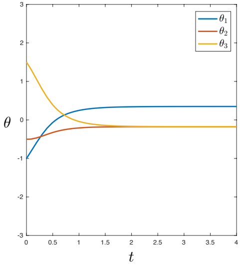
Transition to incoherency

Q1: Existence of a sync state:

$$\dot{\theta}_i = p_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



$a_{ij} = 1$



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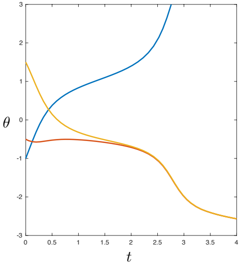

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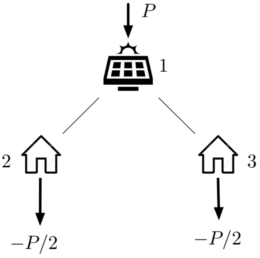

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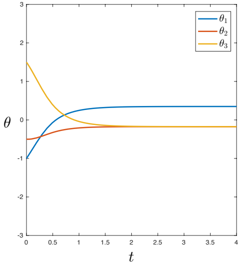

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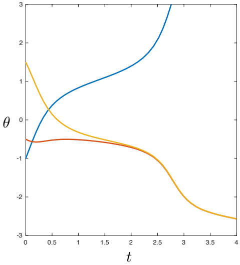

$P = 2.5$

- sync threshold : "power transmission" vs. "coupling"
- quantify: "power transmission" < "coupling"
- as a function of network parameters

Primer on algebraic graph theory

Weighted undirected graph with n nodes and m edges:

Incidence matrix: $n \times m$ matrix B s.t. $(B^\top p)_{(ij)} = p_i - p_j$

Edge weight matrix: $m \times m$ diagonal matrix $\mathcal{A}$

Laplacian matrix: $L = B\mathcal{A}B^\top$

Equilibrium point: $p = B\mathcal{A}\sin(B^\top \theta)$

Algebraic connectivity: $\lambda_2(L) =$ second smallest eig of L

Cycle space: $\mathrm{Ker}(B) =$ span of all the cycle vectors

Known results

Given a network and p, does there exist angles?

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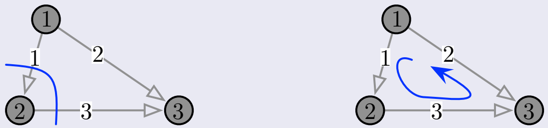
$\|B^\top L^\dagger p\|_\infty < 1$		for trees, complete	(Old ∞-norm T)

Novel: algebraic potential theory



$$\underbrace{\mathbb{R}^m}_{\text{edge space}} = \underbrace{\mathrm{Im}(B^\top)}_{\text{cutset space}} \oplus \underbrace{\mathrm{Ker}(B\mathcal{A})}_{\text{weighted cycle space}}$$

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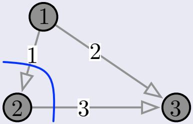

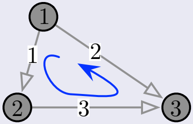

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$$\underbrace{\mathcal{P} = B^\top L^\dagger B\mathcal{A}}_{\text{cutset projection}} = \text{oblique projection onto } \mathrm{Im}(B^\top)$$

parallel to $\mathrm{Ker}(B\mathcal{A})$

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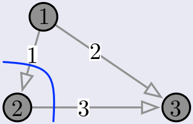

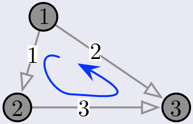

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1. if G acyclic, then $\mathcal{P} = I_m$
2. if G unweighted, then $\mathcal{P}$ is an orthogonal projection
3. if $R_{\text{eff}} \in \mathbb{R}^{n\times n}$ are effective resistances, then $\mathcal{P} = -\frac{1}{2} B^\top R_{\text{eff}} B\mathcal{A}$

Rewriting the equilibrium equation

Find sufficient conditions on $B, \mathcal{A}, p$ s.t. there exists a solution θ to:

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Key idea: Node vs. Edge

$$p = B\mathcal{A}\sin(B^\top \theta) \qquad \textbf{Node balance eq. } \mathbb{R}^n$$

$$\Updownarrow$$

$$B^\top L^\dagger p = \mathcal{P}\sin(B^\top \theta) \qquad \textbf{Edge balance eq. } \mathbb{R}^m$$

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- **Edge variables**: $x = B^\top\theta$ and $z = B^\top L^\dagger p$

Find sufficient conditions on $z \in \operatorname{Im}(B^\top)$ s.t. there exists solution x to:

$$z = \mathcal{P}\sin(x) = \mathcal{P}[\operatorname{sinc}(x)]x$$

2. look for $x \in \mathcal{B}_p(\gamma) = \{x \mid \|x\|_p \leq \gamma\}$ solving

$$\mathcal{P}[\mathrm{sinc}(x)]x = z \quad \iff \quad x = (\mathcal{P}[\mathrm{sinc}(x)])^{-1} z =: h(x)$$

Brouwer's Fixed-Point: A unifying theorem

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$$\|z\|_p \leq \gamma \alpha_p(\gamma) \quad \Longrightarrow \quad h \text{ satisfies Brouwer on } \mathcal{B}_p(\gamma)$$

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Equilibrium angles (neighbors within γ arc) exist if, in some p-norm,

$$\|B^\top L^\dagger p\|_p \leq \gamma \alpha_p(\gamma) \qquad \text{for all graphs} \qquad \text{(New } p\text{-norm T)}$$

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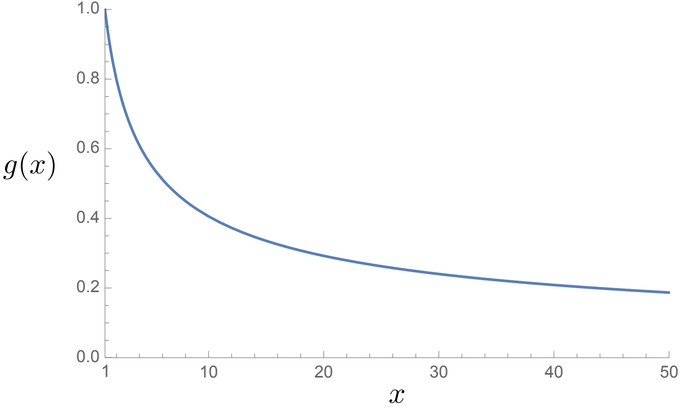
For $p = \infty$, the new test for all graphs

$$\|B^\top L^\dagger p\|_\infty \leq g(\|\mathcal{P}\|_\infty) \qquad \text{(New } \infty\text{-norm T)}$$

Function g is strictly decreasing

$$g : [1, \infty) \to [0, 1]$$



$$g(x) = \frac{y(x) + \sin(y(x))}{2} - x\frac{y(x) - \sin(y(x))}{2}\Big|_{y(x)=\arccos(\frac{x-1}{x+1})}$$

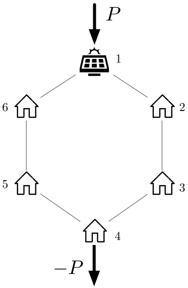
K_{C} = critical coupling of Kuramoto model, computed via MATLAB *fsolve*
K_{T} = smallest value of scaling factor for which test T fails

Test Case	Critical ratio $K_{\mathrm{T}}/K_{\mathrm{C}}$			
	Old 2-norm T	New ∞-norm T $g(\lVert\mathcal{P}\rVert_\infty)$	Old ∞-norm T Approx.test	New ∞-norm T $\alpha_\infty(\pi/2)$
IEEE 9	16.54 %	73.74 %	92.13 %	85.06 %†
IEEE RTS 24	3.86 %	53.44 %	89.48 %	89.48 %†
New England 39-bus	2.97 %	67.57 %	100 %	100 %†
IEEE 118	0.29 %	43.70 %	85.95 %	—*
IEEE 300	0.20 %	40.33 %	99.80 %	—*
Polish 2383	0.11 %	29.08 %	82.85 %	—*

† *fmincon* has been run for 100 randomized initial phase angles.
* *fmincon* does not converge.

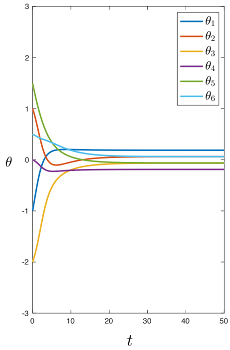
Q2: Is the equilibrium point unique?

$$\dot{\theta}_i = p_i - \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$

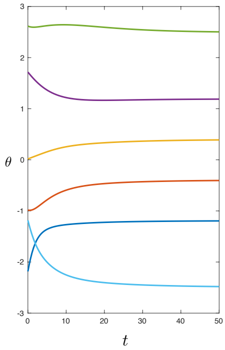


$a_{ij} = 1$

$P = 1/4$



$\theta_0 = [-1, 1, -2, 0, 1.5, 0.5]^\top$



$\theta_0 = [0, 1.2, 2.2, 3.9, 4.8, 1]^\top$

Q2: Is the equilibrium point unique?

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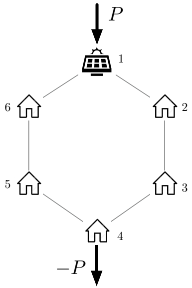

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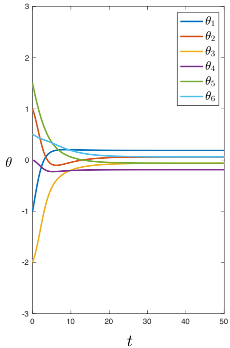

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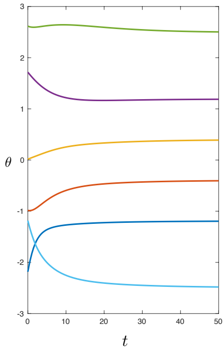

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- multistable sync : "cycle structure" and "state space"
- quantify: "cycle structure" vs "multistable sync"

Algebraic graph theory on n-torus

Key question

How to localize stable equilibrium points?

Winding number

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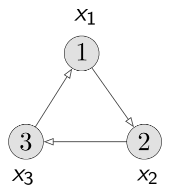
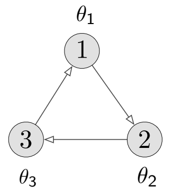
Nodal variables in $\mathbb{R}^3$



$$\sum_{i=1}^{3} \overbrace{(x_i - x_{i+1})}^{\text{distance in } \mathbb{R}} = 0.$$

Nodal variables in $\mathbb{T}^3$



$$\sum_{i=1}^{3} \overbrace{(\theta_i - \theta_{i+1})}^{\text{distance in } \mathbb{S}} = 2\pi w_\sigma(\theta),$$

$w_\sigma(\theta) \in \mathbb{Z}$, winding number

Winding partition of the n-torus

Winding vector

Given a graph G with a cycle basis $\Sigma = \{\sigma_1, \dots, \sigma_{m-n+1}\}$ and $\theta \in \mathbb{T}^n$.

Winding vector: $\mathbf{w}_\Sigma(\theta) = [w_{\sigma_1}(\theta), \dots, w_{\sigma_{m-n+1}}(\theta)]^\top \in \mathbb{Z}^{m-n+1}$

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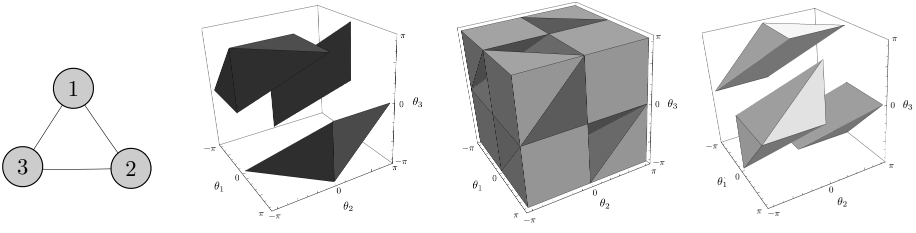
Winding cells: equivalence classes

Given a graph G with a cycle basis Σ. For every $\mathbf{u} \in \mathbb{Z}^{m-n+1}$

$$(\text{Winding cell } \mathbf{u}) \quad = \quad \text{all } \theta \in \mathbb{T}^n \text{ s.t. } \mathbf{w}_\Sigma(\theta) = \mathbf{u}.$$



$\mathbf{u} = -1$ $\qquad$ $\mathbf{u} = 0$ $\qquad$ $\mathbf{u} = +1$

Winding partition of n-torus

$\mathbb{T}^n = \bigcup_{\mathbf{u}\in\mathbb{Z}^{m-n+1}} (\text{Winding cell } \mathbf{u})$

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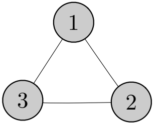

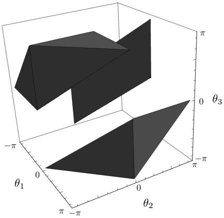

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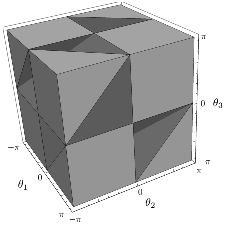

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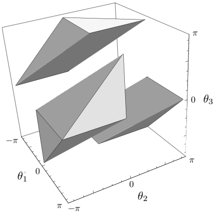

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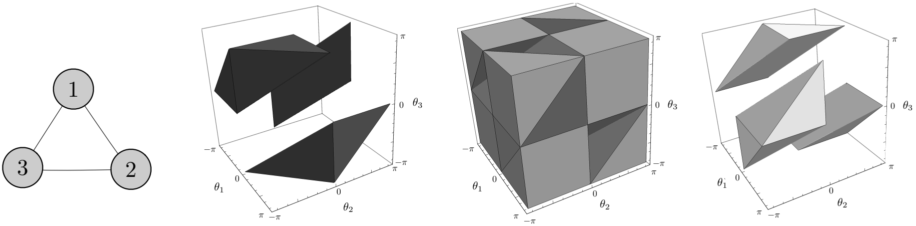
$$p_i = \sum_{j=1}^{n} a_{ij} \sin(\theta_i - \theta_j)$$



$\mathbf{u} = -1$ $\mathbf{u} = 0$ $\mathbf{u} = +1$

At-most uniqueness

There is either **zero** or **one** stable equilibrium point (neighbors within $\pi/2$ arc) in each winding cell.

Existence/Computation

Key question

How to check if we have a stable equilibrium point inside a winding cell?

Existence/Computation

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How to check if we have a stable equilibrium point inside a winding cell?

Nodal Balance

$$\begin{cases} p = B\mathcal{A}\sin(B^\top\theta), \\ \text{neighbors within } \gamma \text{ arc} \end{cases}$$

Edge Balance

$$\begin{cases} B^\top L^\dagger p = \mathcal{P}_{\mathrm{cut}}\eta, & \|\eta\|_\infty \leq \sin(\gamma), \\ \mathcal{P}_{\mathrm{cyc}}(\arcsin(\eta) - 2\pi C_\Sigma^\dagger \mathbf{u}) = \mathbb{0}_m. \end{cases}$$

- $\mathcal{P}_{\mathrm{cut}}$ is the projection onto the cutset space
- $\mathcal{P}_{\mathrm{cyc}}$ is the projection onto the cycle space
- Separates the cycle flows and cutset flows
- Highlights the role of the winding vector

Existence/Computation

Iterations for edge balance equations

$$\eta^{(k+1)} = B^\top L^\dagger p + \mathcal{P}_{\mathrm{cyc}}\Big(\eta^{(k)} - \cos(\gamma)\big(\mathrm{asin}_\gamma(\eta^{(k)}) - 2\pi C_\Sigma^\dagger \mathbf{u}\big)\Big).$$

- Start from any $\eta^{(0)} \in \mathbb{R}^m$
- The sequence is **contractive** and always converges (to a vector η^*)
- **If** $\|\eta^*\|_\infty > \sin(\gamma)$: no stable equilibrium point in winding cell u.
- **If** $\|\eta^*\|_\infty \leq \sin(\gamma)$: one stable equilibrium point in winding cell u;

$$\theta^* = L^\dagger B \mathcal{A}(\mathrm{asin}(\eta^*) - 2\pi C_\Sigma^\dagger \mathbf{u})$$

Contributions

- geometry of cutset projection operator
- family of sufficient sync conditions
- partiton of n-torus based on winding vector
- localize the equilibrium points using winding cells

Summary and future work

Contributions

- geometry of cutset projection operator
- family of sufficient sync conditions
- partiton of n-torus based on winding vector
- localize the equilibrium points using winding cells

Future research

- close the gap between sufficient and necessary conditions
- region of attraction of stable equilibrium points
- generalizations to other oscillator models.