Resilience of Input Metering in Dynamic Flow Networks

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Introduction: Flow Networks

Definition and Examples

- Flow of a commodity through the network

commodity=electric power

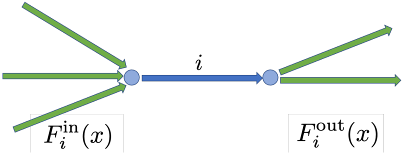
commodity=vehicles

commodity=water

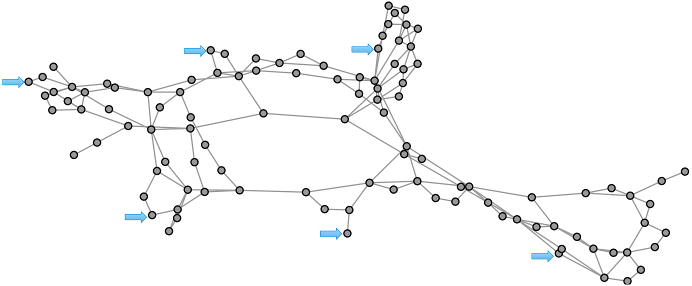
x_i = density of the commodity at ith compartment

$$\dot{x}_i = F_i^{\text{in}}(x) - F_i^{\text{out}}(x),$$

$F_i^{\text{in}}(x)$, $F_i^{\text{out}}(x)$: inflow and outflow to compartment i



Design suitable input strategy that perform a desired task



Input metering strategy are mostly designed for a **nominal** setting

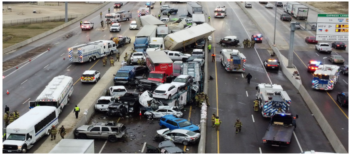
- **Traffic networks**: ramp metering (in practice often not closed-loop)

Transient Perturbations

Examples and Effect on Input Metering

Uncertainties can compromise the performance of input metering strategies

1. **Traffic networks**:
   - weather condition, driving behavior
   - reduced road capacity and congestion
2. **Power grids**:
   - line tripping, consumption patterns
   - blackouts and cascading failures
3. **Water networks**:
   - precipitation, pipe leakage
   - water supply disruption



Challenge

Robustness of the input metering strategy wrt to transient perturbations

Dynamic Flow Networks

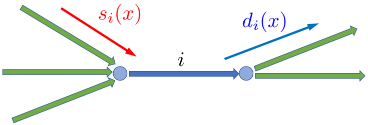
Modeling: supply and demand functions

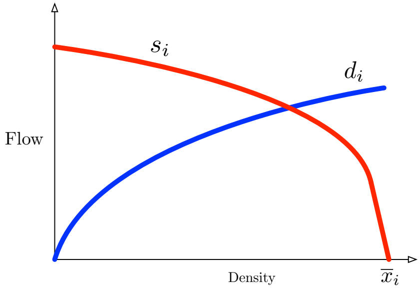
- network of $\mathcal{L}$ compartments
- commodity flows from compartment to compartment
- some compartments $\mathcal{R} \subset \mathcal{L}$ take input from the environment
- density in compartment i is $x_i \in [0, \overline{x}_i]$ with capacity $\overline{x}_i$.

$$\dot{x}_i = F_i^{\text{in}}(x, u) - F_i^{\text{out}}(x) := f_i(x, u),$$

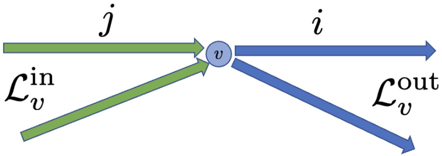
Demand and supply of compartment i

(demand)	$x_i \mapsto d_i(x_i)$	increasing
(supply)	$x_i \mapsto s_i(x_i)$	decreasing





$$\mathcal{L} = \overbrace{\mathcal{R}}^{\text{inputs}} \cup \overbrace{\mathcal{O}}^{\text{others}}$$



Input metering

$$F_i^{\text{in}}(x) = \min\{u_i, s_i(x_i)\}$$

Fixed routing ratios

$$F_i^{\text{in}}(x) = R_i^v \sum_{j\in\mathcal{L}_v^{\text{in}}} F_j^{\text{out}}(x_j)$$

Conservation of flow

$$\sum_{i\in\mathcal{L}_v^{\text{out}}} R_i^v \leq 1$$

First-In-First-Out (FIFO rule)

$$F_j^{\text{out}}(x) = \alpha^v(x) d_j(x_j),$$

$$\alpha^v(x) = \min_{l\in\mathcal{L}_v^{\text{out}}} \left\{1, \frac{s_l(x_l)}{R_l^v \sum_{k\in\mathcal{L}_v^{\text{in}}} d_k(x_k)}\right\}$$

- Compartment j is in **congestion** if $\alpha^v(x) < 1$
- Compartment j is in **free-flow** if $\alpha^v(x) = 1$

Input Metering

Resilience wrt to transient perturbations

Given an input metering strategy u:

Before perturbation

$$\dot{x}_i = F_i^{\text{in}}(x,u) - F_i^{\text{out}}(x)$$

$$0 \to t_0$$

During perturbation

$$\dot{x}_i = F_i^{\text{in}}(x,u) - F_i^{\text{out}}(x) + \delta_i(x)$$

$$t_0 \to t_1$$

After perturbation

$$\dot{x}_i = F_i^{\text{in}}(x,u) - F_i^{\text{out}}(x)$$

$$t_1 \to \infty$$

$\delta(x) = [\delta_1(x), \ldots, \delta_n(x)]^\top$ is an arbitrary transient perturbation

Main question

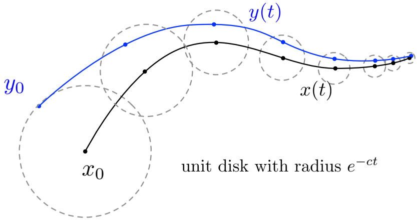
Can the system recover from the transient perturbations?

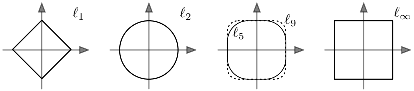
Problem Statement

Is $x(t_1)$ in the **region of attraction (ROA)** of the **desirable operating point** of the flow network?

Dynamical system $\dot{x} = \mathsf{G}(x)$ on $\mathbb{R}^n$ is

- **contractive** if its flow is a contracting map
- **weakly-contractive** if its flow is a non-expansive map
- **monotone** if its flow preserves the partial ordering $\leq$ on $\mathbb{R}^n$

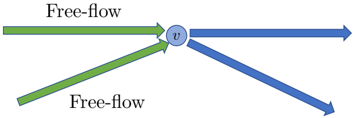




Monotone domain

$$\mathcal{M} = \{x \in [\mathbb{0}_{\mathcal{L}}, \overline{x}] \mid F_i^{\text{out}}(x) = d_i(x_i), \text{ for } i \in \mathcal{L}_v^{\text{in}} \text{ with } v \text{ div. junction } \}.$$

- **Intuition:** the upstreams of diverging junctions are in *free-flow* inside $\mathcal{M}$



Theorem: flow networks on monotone domain

For input metering u, flow network on $\mathcal{M}$

1. is monotone = the commodities preserve the partial ordering
2. is weakly-contracting wrt ℓ_1-norm = $\sum_{i=1}^{|\mathcal{L}|} |x_i|$ is non-increasing
3. has a unique free-flow equilibrium point $x^*(u)$ if $u \in \mathcal{U}^{\text{feasible}}$;

Theorem: dichotomy for asymptotic behaviors

For a weakly-contracting system $\dot{x} = \mathsf{G}(x)$, either

1. G has no equilibrium and every trajectory is unbounded, or
2. G has at least one equilibrium x^* and every trajectory is bounded,
   - if the norm $\|\cdot\|$ is a p-norm, $p \in \{1, \infty\}$ and G is piecewise real analytic, then every trajectory converges to the set of equilibria,

Corollary

If $u \in \mathcal{U}^{\text{feasible}}$, then any invariant set in $\mathcal{M}$ is inside ROA of $x^*(u)$

S. Jafarpour, P. Cisneros-Velarde, and F. Bullo. Weak and semi-contraction for network systems and diffusively-coupled oscillators. *IEEE Transactions on Automatic Control*, Feb. 2021

Key idea

Monotone extension h of f obtained by:

$$\alpha^v(x) \mapsto \beta^v(x) = \begin{cases} \alpha^v(x) & x \in \mathcal{M}, \\ 1 & x \notin \mathcal{M}. \end{cases}$$

$$f(x,u) \mapsto h(x,u)$$

- h is monotone on $[\mathbb{0}_{|\mathcal{L}|}, \overline{x}]$
- h is weakly-contracting on $[\mathbb{0}_{|\mathcal{L}|}, \overline{x}]$
- $f(x,u) = h(x,u)$ for every $x \in \mathcal{M}$

Theorem: ROA of flow networks

Let $t \mapsto y(t)$ be the solution to

$$\dot{y} = h(y,u),$$
$$y(0) = \overline{x}$$

and let $t^* = \min\{t \in \mathbb{R}_{\geq 0} \mid y(t) \in \mathcal{M}\}$

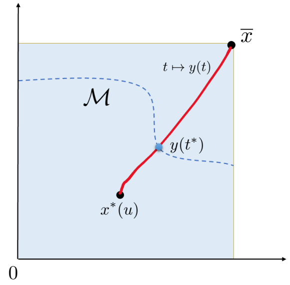
1. $\lim_{t\to\infty} y(t) = x^*(u)$
2. $[\mathbb{0}_{|\mathcal{L}|}, y(t^*)]$ is an invariant set in $\mathcal{M}$
3. every trajectory of f starting from $[\mathbb{0}_{|\mathcal{L}|}, y(t^*)]$ converges to $x^*(u)$.

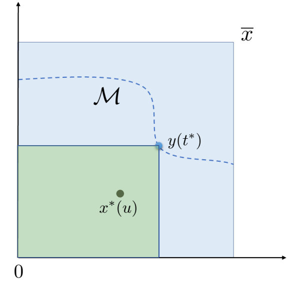
Corollary

For networks with no div. junction, $x^*(u)$ is globally stable

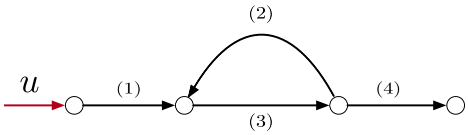
Sketch of the proof

1. red line: trajectory of monotone extension h starting from $\overline{x}$
2. **weakly-contractive**: the trajectory converges to $x^*(u)$
3. the trajectory crosses the boundary of $\mathcal{M}$ at $y(t^*)$
4. **monotone**: the box $[\mathbb{0}, y(t^*)]$ is an invariant set





Example: Cyclic Flow Network



For $i \in \{1, \ldots, 4\}$

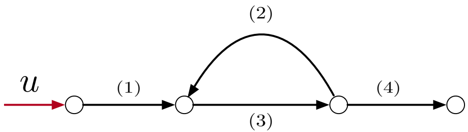
$$x_i \in [0, 30]$$
$$d_i(x_i) = x_i$$
$$s_i(x_i) = 30 - x_i$$

routing ratios

$$R_i^v = \begin{cases} \frac{1}{2} & i = 2, 4, \\ 1 & i = 1, 3 \end{cases}$$

Dynamics of the system

$$\dot{x}_1 = \min\{u, 30 - x_1\} - x_1 \min\{1, \tfrac{30-x_3}{x_1+x_2}\},$$
$$\dot{x}_2 = \tfrac{1}{2}\min\{x_3, \tfrac{30-x_2}{2}, \tfrac{30-x_4}{2}\} - x_2 \min\{1, \tfrac{30-x_3}{x_1+x_2}\}$$
$$\dot{x}_3 = \min\{x_1 + x_2, 30 - x_3\} - \min\{x_3, \tfrac{30-x_2}{2}, \tfrac{30-x_4}{2}\}$$
$$\dot{x}_4 = \tfrac{1}{2}\min\{x_3, \tfrac{30-x_2}{2}, \tfrac{30-x_4}{2}\} - x_4$$



Equilibrium points

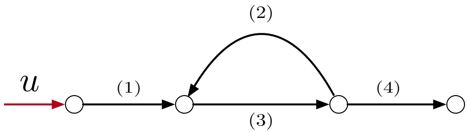
Input metering $u = 5$, **two** stable equilibria

- free-flow equilibrium $x^* = (5,\ 5,\ 10,\ 5)^\top$
- non-free-flow equilibrium $x^{**} = (30,\ 30,\ 30,\ 0)^\top$

The monotone domain:

$$\mathcal{M} = \{x \in \mathbb{R}^4 \mid x_3 \leq \min\{\tfrac{30-x_2}{2}, \tfrac{30-x_4}{2}\}\}$$

Example: Cyclic Flow Network



Given the Input metering $u = 5$,

ROA of x^*

- $t \mapsto y(t)$ solution of monotone extension starting from $(30,\ 30,\ 30,\ 30)^\top$.
- at $t^* = 11.465$ we have $y(t^*) = (15,\ 22.5,\ 15,\ 7.5)^\top \in \mathcal{M}$
- $[\mathbb{0}_4, (15,\ 22.5,\ 15,\ 7.5)^\top]$ is in the region of attraction of x^*

ROA of x^{**}

Starting from $x = (20,\ 20,\ 15,\ 5)^\top$, system converges to x^{**}

Conclusions

- introduced dynamical flow networks
- region of attraction to understand the effect of transient perturbations
- weak-contractivity and monotonicity of dynamical flow networks
- under-approximate the region of attraction

Thank you for your attention!