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Improving the precision of noisy oscillators

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HIGHLIGHTS

- Derives a general formula for the precision of an oscillator affected by white noise.
- Performs detailed calculations for illustrative examples.
- Demonstrates how to improve a noisy oscillator's precision by appropriately tuning system parameters.
- Demonstrates how to improve a noisy oscillator's precision through appropriately timed impulsive kicks.
- Interprets the results in terms of isochrons and phase response curves.

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ABSTRACT

We consider how the period of an oscillator is affected by white noise, with special attention given to the cases of additive noise and parameter fluctuations. Our treatment is based upon the concepts of isochrons, which extend the notion of the phase of a stable periodic orbit to the basin of attraction of the periodic orbit, and phase response curves, which can be used to understand the geometry of isochrons near the periodic orbit. This includes a derivation of the leading-order effect of noise on the statistics of an oscillator's period. Several examples are considered in detail, which illustrate the use and validity of the theory, and demonstrate how to improve a noisy oscillator's precision by appropriately tuning system parameters or operating away from a bifurcation point. It is also shown that appropriately timed impulsive kicks can give further improvements to oscillator precision.

1. Introduction

Many physical, biological, and technological systems produce rhythmic oscillations that are self-sustained, with an internal source of energy being transformed into oscillations of chemical concentrations, electrical properties, and/or mechanical properties [1–3]. In the absence of noise and for constant parameters, one expects the period of such oscillations to be perfectly regular, in which case they are said to have perfect precision. However, real systems are subject to noise and fluctuating parameters, and, in general, such stochastic effects will degrade the precision of the oscillations. Such degradation can decrease the performance of technological devices [4–6], for example reducing the accuracy in determining relative distances between objects, reducing the density of frequencies that can be resolved from a radio spectrum, and hindering the ability to detect chemical or biological agents. With this in mind, we seek to understand and improve the precision of oscillators that operate in noisy conditions.

Our treatment is based upon the mathematical technique of phase reduction, and in particular will use the concepts of isochrons [7] and phase response curves [8]. Isochrons are foliations of phase space that extend the notion of the phase of a stable periodic orbit to the basin of attraction of the periodic orbit. Each point in the basin of attraction lies on only one isochron, and two points on the same isochron converge to the periodic orbit with the same phase. As we will see, isochrons provide the natural coordinate system for understanding and calculating the precision of the period for an oscillator. Phase response curves, which give the change in phase associated with an impulsive perturbation, are useful here because they can be used to understand the geometry of isochrons near the periodic orbit. We note that some of our calculations provide alternative derivations for results given in [9], here using the language of isochrons and phase response curves. A feature of our treatment is that it allows one to make several interesting connections with results from mathematical neuroscience and systems biology, and to understand how the geometry of the isochrons affects oscillator precision.

To set up our general discussion, it is instructive to start with some simple, relevant ideas. First, suppose that at a particular time, a perturbation $\delta\mathbf{x}$ is made to the state $\mathbf{x}$ of a system so that $\mathbf{x} \to \mathbf{x} + \delta\mathbf{x}$. If $\delta\mathbf{x}$ is such that the system stays on the same isochron,

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that is, if $\theta(\mathbf{x}) = \theta(\mathbf{x} + \delta\mathbf{x})$, where θ is the phase associated with the state as defined by the isochrons, then the perturbation does not affect the oscillator's phase, and the period of the oscillation, defined as the time it takes for the oscillator to return to its phase before the perturbation, will not change. If, on the other hand, the perturbation moves the system to a different isochron, then the oscillator has moved to a different phase, and there will be a change in its period resulting in a loss of precision. Our results will characterize this loss of precision for oscillators which receive such perturbations due to noise, and show how it can be reduced by tuning system parameters and with appropriate impulsive kicks.

We are also interested in understanding how the fluctuation of a system parameter can affect the precision of an oscillator. Suppose that a parameter value changes by a small amount, small enough that the periodic orbit continues to exist. In general, the periodic orbit for the new parameter value will not be identical to the periodic orbit for the original parameter value; there could be a change in the "shape" of the periodic orbit, and/or a change in its period. Furthermore, the isochrons for the periodic orbit for the new parameter value will in general not be identical to the isochrons for the periodic orbit for the original parameter value. However, suppose that a system is designed so that changing this parameter only affects the dynamics in the direction of the isochrons. If such a parameter rapidly fluctuates about a constant value, it will not affect the phase of the oscillator, and therefore there will be no loss of precision associated with the fluctuating parameter. One intuitively simple way to accomplish this is to design an oscillator with the following characteristics: (i) the (stable) periodic orbit is a circle; (ii) the isochrons in the neighborhood of this periodic orbit are radial; (iii) the fluctuating parameter only affects the dynamics in the radial direction. An example will be presented in Section 4.1 which satisfies these characteristics, and our treatment will show how to generalize this concept.

Finally, we will show how impulsive kicks can be used to improve oscillator precision. The main idea here is to kick the system once per period into a region of phase space where it is less sensitive to noise, without introducing significant imprecision due to the kicks themselves. The regions of phase space where noise has a smaller effect will be those for which the isochrons are more "diffuse"; then the same perturbations from the noise will lead to a relatively smaller change of phase.

This paper is organized as follows. In Section 2, we give a formal definition of isochrons, and describe two common methods for calculating phase response curves. In Section 3, we derive the effect of white noise on an oscillator's period, giving formulas for the mean and standard deviation of the period to leading order in the strength of the noise. Section 4 considers how to improve oscillator precision for vector fields for which the dynamics do not explicitly depend on the phase. Section 5 investigates how noise affects the period for a Hindmarsh–Rose neuron. Section 6 describes how impulsive kicks can be used to improve oscillator precision. Finally, Section 7 summarizes the results, and discusses possible extensions of these ideas.

2. Isochrons

2.1. Definition of isochrons

We begin by defining isochrons precisely. Consider an autonomous vector field

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}), \quad \mathbf{x} \in \mathbb{R}^n, \quad (n \geq 2) \tag{1}$$

having a stable hyperbolic periodic orbit $\mathbf{x}^\gamma(t)$ with period T. For each point $\mathbf{x}^*$ in the basin of attraction of the periodic orbit, there exists a unique $\theta(\mathbf{x}^*)$ such that

$$\lim_{t\to\infty} \left| \mathbf{x}(t) - \mathbf{x}^\gamma \left(t + \frac{T}{2\pi}\, \theta(\mathbf{x}^*) \right) \right| = 0, \tag{2}$$

where $\mathbf{x}(t)$ is a trajectory starting with the initial point $\mathbf{x}^*$. The function $\theta(\mathbf{x})$ is called the *asymptotic phase* of $\mathbf{x}$, and takes values in $[0, 2\pi)$. (Other conventions, related to this through a simple rescaling, define the asymptotic phase to take values in $[0, T)$ or in $[0, 1)$.) An *isochron* is a level set of $\theta(\mathbf{x})$, that is, the collection of all points in the basin of attraction of $\mathbf{x}^\gamma$ with the same asymptotic phase. Isochrons extend the notion of phase of a stable periodic orbit to the basin of attraction of the periodic orbit. It is conventional to define isochrons so that the phase of a trajectory on the periodic orbit advances linearly in time, so that

$$\frac{d\theta}{dt} = \frac{2\pi}{T} \equiv \omega \tag{3}$$

both on and off the periodic orbit. Points at which isochrons of a periodic orbit cannot be defined form the *phaseless set*.

Isochrons can be shown to exist for any stable hyperbolic periodic orbit. They are codimension one manifolds as smooth as the vector field, and transversal to the periodic orbit $\mathbf{x}^\gamma$. Their union covers an open neighborhood of $\mathbf{x}^\gamma$. This can be proved directly by using the Implicit Function Theorem [10,11], and is also a consequence of results on normally hyperbolic invariant manifolds [12].

2.2. Obtaining local approximations to isochrons

We now briefly describe two methods for obtaining a local approximation to the isochron passing through the base point $\tilde{\mathbf{x}}^\gamma$ on the periodic orbit. These calculate the gradient $\nabla\theta$ of the asymptotic phase at the base point; this gradient is typically referred to as the (infinitesimal) *phase response curve*, and can be interpreted as capturing the effect of small impulsive perturbations to a system. Isochrons, being sets of constant asymptotic phase, are tangent to the $(n-1)$-dimensional planes normal to this gradient.

We note that for some systems it is possible to obtain analytical expressions for local approximations to isochrons [13–17]. Moreover, there are other methods for calculating isochrons in the neighborhood of a periodic orbit, for example [18–21], plus methods which can accurately find them for the whole basin of attraction of the periodic orbit [22,23]. See [7] for more discussion about isochrons.

2.2.1. The direct method

Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$. By definition

$$\frac{\partial\theta}{\partial x_i}\left(\tilde{\mathbf{x}}^\gamma\right) = \lim_{\Delta x_i\to 0} \frac{\Delta\theta}{\Delta x_i}, \tag{4}$$

where $\Delta\theta = \left[\theta(\tilde{\mathbf{x}}^\gamma + \Delta x_i \hat{i}) - \theta(\tilde{\mathbf{x}}^\gamma)\right]$ is the change in $\theta(\mathbf{x})$ resulting from a perturbation $x_i \to x_i + \Delta x_i$ from the base point $\tilde{\mathbf{x}}^\gamma$ in the direction of the ith coordinate. Since (3) holds everywhere in the neighborhood of $\mathbf{x}^\gamma$, the difference $\Delta\theta$ is preserved under the flow; thus, it may be measured in the limit as $t \to \infty$, when the perturbed trajectory has collapsed back to the periodic orbit. That is, $\frac{\partial\theta}{\partial x_i}\left(\tilde{\mathbf{x}}^\gamma\right)$ can be found by comparing the phases of solutions in the infinite-time limit starting on and infinitesimally shifted from base points on the periodic orbit [24,25,16]. This method is commonly used to experimentally approximate the phase response curve for an oscillator, e.g., [26].

2.2.2. The adjoint method

As shown, for example, in Appendix A of [16], the phase response curve can also be found by solving the following adjoint equation:

$$\frac{d\nabla_{\mathbf{x}^\gamma(t)}\theta}{dt} = -D\mathbf{F}^T(\mathbf{x}^\gamma(t))\,\nabla_{\mathbf{x}^\gamma(t)}\theta, \tag{5}$$

subject to the condition

$$\nabla_{\mathbf{x}^\gamma(0)}\theta\cdot\mathbf{F}(\mathbf{x}^\gamma(0)) = \omega \tag{6}$$

and requiring that the solution $\nabla_{\mathbf{x}^\gamma(t)}\theta$ to (5) is T-periodic.

3. The effect of noise on an oscillator's period

Here we consider the effect of Gaussian white noise processes on the period of an oscillator. Our calculation adapts the arguments in [27]; similar results have been obtained by other methods in [9].

Consider the general stochastic differential equation

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) + \sigma\mathbf{B}(\mathbf{x})\eta(t), \tag{7}$$

where $\mathbf{x}\in\mathbb{R}^n$, $\eta\in\mathbb{R}^m$, and $\mathbf{B}$ is a real $n\times m$ matrix. Here $\eta(t)$ is a vector of real, independent Gaussian white noise random processes with the properties

$$\langle\eta_i(t)\rangle = 0,\quad \langle\eta_i(t)\eta_j(t')\rangle = \delta_{ij}\delta(t-t'). \tag{8}$$

We assume that when $\sigma = 0$, this system has a stable hyperbolic periodic orbit $\mathbf{x}^\gamma(t)$. Eq. (7) can be rewritten as

$$d\mathbf{x} = \mathbf{F}(\mathbf{x})dt + \sigma\mathbf{B}(\mathbf{x})d\mathbf{W}(t) \tag{9}$$

where $d\mathbf{W}(t) = \eta(t)dt$ and $\mathbf{W}(t)$ is a vector of independent standard Wiener processes. Here we are thinking of σ as the strength of the largest noise term; the other noise strengths can be incorporated into $\mathbf{B}$ as appropriate. For simplicity, in the following we will consider small noise, only keeping terms up to $\mathcal{O}(\sigma)$; higher order terms can be found by adapting the calculations in [27]. Letting $\theta=\theta(\mathbf{x})$ and applying Ito's formula [28] to leading order in σ,

$$d\theta = \sum_{i=1}^n \frac{\partial\theta}{\partial x_i}F_i(\mathbf{x})dt + \sigma\sum_{i=1}^n\sum_{j=1}^m \frac{\partial\theta}{\partial x_i}B_{ij}(\mathbf{x})dW_j(t) + \mathcal{O}(\sigma^2). \tag{10}$$

Evaluating this on the periodic orbit and defining the ith component of the (infinitesimal) phase response curve as

$$\left.\frac{\partial\theta}{\partial x_i}\right|_{\mathbf{x}^\gamma(t)} \equiv Z_i(\theta), \tag{11}$$

we obtain

$$d\theta = \omega dt + \sigma\sum_{i=1}^n\sum_{j=1}^m Z_i(\theta)B_{ij}(\theta)dW_j(t) + \mathcal{O}(\sigma^2). \tag{12}$$

Note that in this expression, B_{ij} is evaluated on the periodic orbit which exists for $\sigma = 0$, and is rewritten to be a function of the phase variable θ.

It is important to mention that care must be taken if we are interested in the $\mathcal{O}(\sigma^2)$ terms for such a phase reduction in the presence of noise. In particular, the $\mathcal{O}(\sigma^2)$ terms which appear in (12) will depend on the relationship between the timescale at which the amplitude variable relaxes to the limit cycle and the correlation time of the noise [29], cf. [30]. Our calculations will all be done to $\mathcal{O}(\sigma)$, so we do not have to consider these subtleties here.

Taking $\theta(0)=0$, Eq. (12) has a solution

$$\theta(t) = \omega t + \sigma\sum_{i=1}^n\sum_{j=1}^m\int_0^t Z_i(\theta(s))B_{ij}(\theta(s))dW_j(s) + \cdots. \tag{13}$$

To leading order,

$$Z_i(\theta(s)) = Z_i(\omega s + \cdots) = Z_i(\omega s) + \cdots,$$
$$B_{ij}(\theta(s)) = B_{ij}(\omega s + \cdots) = B_{ij}(\omega s) + \cdots,$$

giving

$$\theta(t) = \omega t + \sigma\sum_{i=1}^n\sum_{j=1}^m\int_0^t Z_i(\omega s)B_{ij}(\omega s)dW_j(s) + \cdots. \tag{14}$$

We now expand the period T in powers of σ:

$$T = \frac{2\pi}{\omega} + \sigma T_1 + \cdots. \tag{15}$$

Plugging this into (14), and using the fact that $\theta(T) = 2\pi$ by definition,

$$\theta(T) = 2\pi = \omega T + \sigma Y(T) + \cdots \tag{16}$$
$$= \omega\left(\frac{2\pi}{\omega} + \sigma T_1 + \cdots\right) + \sigma Y(T) + \cdots \tag{17}$$
$$= 2\pi + \sigma(\omega T_1 + Y(T)) + \cdots, \tag{18}$$

where

$$Y(T) \equiv \sum_{i=1}^n\sum_{j=1}^m\int_0^T Z_i(\omega s)B_{ij}(\omega s)dW_j(s). \tag{19}$$

For this to be valid, the term multiplying σ must vanish. Therefore,

$$T_1 = -\frac{1}{\omega}Y(T)$$
$$= -\frac{1}{\omega}\sum_{i=1}^n\sum_{j=1}^m\int_0^{2\pi/\omega} Z_i(\omega s)B_{ij}(\omega s)dW_j(s) + \cdots. \tag{20}$$

From (4.2.40) of [28], $\langle T_1\rangle = 0$, so

$$\langle T\rangle = \frac{2\pi}{\omega} + \mathcal{O}(\sigma^2). \tag{21}$$

Moreover,

$$\begin{aligned}
&\mathrm{stdev}(T)\\
&= \langle(T-\langle T\rangle)^2\rangle^{1/2} = \sigma\langle T_1^2\rangle^{1/2} + \cdots\\
&= \frac{\sigma}{\omega}\left\langle\int_0^{2\pi/\omega}\sum_{i=1}^n\sum_{j=1}^m Z_i(\omega s)B_{ij}(\omega s)dW_j(s)\right.\\
&\quad\times\left.\int_0^{2\pi/\omega}\sum_{k=1}^n\sum_{l=1}^m Z_k(\omega s)B_{kl}(\omega s)dW_l(s)\right\rangle^{1/2} + \cdots\\
&= \frac{\sigma}{\omega}\left(\sum_{i=1}^n\sum_{j=1}^m\sum_{k=1}^n\sum_{l=1}^m\left\langle\int_0^{2\pi/\omega} Z_i(\omega s)B_{ij}(\omega s)dW_j(s)\right.\right.\\
&\quad\times\left.\left.\int_0^{2\pi/\omega} Z_k(\omega s)B_{kl}(\omega s)dW_l(s)\right\rangle\right)^{1/2} + \cdots\\
&= \frac{\sigma}{\omega}\left(\sum_{i=1}^n\sum_{j=1}^m\sum_{k=1}^n\int_0^{2\pi/\omega} Z_i(\omega s)B_{ij}(\omega s)Z_k(\omega s)B_{kj}(\omega s)ds\right)^{1/2} \qquad (22)\\
&\quad+ \cdots\\
&= \frac{\sigma}{\omega}\left(\sum_{j=1}^m\int_0^{2\pi/\omega}\sum_{i=1}^n Z_i(\omega s)B_{ij}(\omega s)\sum_{k=1}^n Z_k(\omega s)B_{kj}(\omega s)ds\right)^{1/2}
\end{aligned}$$

$$+\cdots$$

$$= \frac{\sigma}{\omega}\left(\sum_{j=1}^{m}\int_0^{2\pi/\omega}\left(\sum_{k=1}^{n} Z_k(\omega s)B_{kj}(\omega s)\right)^2 ds\right)^{1/2} + \cdots \quad (23)$$

$$= \frac{\sigma}{\omega^{3/2}}\left(\sum_{j=1}^{m}\int_0^{2\pi}\left(\sum_{k=1}^{n} Z_k(\Theta)B_{kj}(\Theta)\right)^2 d\Theta\right)^{1/2} + \cdots. \quad (24)$$

Here (23) follows from (4.2.42) from [28], adapted to allow independent noise sources, and (24) follows from the change of variables $\Theta = \omega s$. Eq. (24) gives a useful formula for the precision of an oscillator.

3.1. Additive Gaussian white noise

As an example, suppose

$$\frac{dx_i}{dt} = F_i(\mathbf{x}) + \sigma_i\eta_i(t), \quad i = 1, \ldots, n. \quad (25)$$

This can be rewritten as

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) + \sigma\mathbf{B}\eta(t), \quad (26)$$

where $B_{ij} = \delta_{ij}\sigma_i/\sigma$, with $\sigma = \max_{i=1}^{n}\sigma_i$. Then Eq. (24) gives

$$\text{stdev}(T) = \frac{1}{\omega^{3/2}}\left(\sum_{j=1}^{n}\sigma_j^2\int_0^{2\pi}[Z_j(\Theta)]^2 d\Theta\right)^{1/2} + \cdots. \quad (27)$$

We see that, to leading order, the phase response curves $Z_j(\Theta), j = 1, \ldots, n$ determine the precision of the oscillator under additive Gaussian white noise, cf. [31]; see also [32,33] which derive related expressions for more general noise.

3.2. Parameter fluctuations

Suppose

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}, \mathbf{p}), \quad \mathbf{x} \in \mathbb{R}^n, \ \mathbf{p} \in \mathbb{R}^q, \quad (28)$$

where

$$p_i = p_{i0} + \sigma_i\eta_i(t), \quad i = 1, \ldots, q. \quad (29)$$

Taylor expanding about $\mathbf{p} = \mathbf{p}_0$, we obtain

$$\frac{d\mathbf{x}}{dt} = \mathbf{F}(\mathbf{x}) + \sigma\mathbf{B}\eta(t), \quad (30)$$

where $B_{ij} = \frac{\sigma_j}{\sigma}\frac{\partial F_i}{\partial p_j}$, and $\sigma = \max_{i=1}^{q}\sigma_i$. Then

$$\text{stdev}(T) = \frac{1}{\omega^{3/2}}\left(\sum_{j=1}^{q}\sigma_j^2\int_0^{2\pi}\left(\sum_{k=1}^{n}\frac{\partial\theta}{\partial x_k}\frac{\partial F_k}{\partial p_j}\right)^2 d\Theta\right)^{1/2} + \cdots. \quad (31)$$

We note that $\sum_{k=1}^{n}\frac{\partial\theta}{\partial x_k}\frac{\partial F_k}{\partial p_j}$ is known in the systems biology literature as the parametric impulsive phase response curve [34], and represents the change in phase associated with the impulsive change dp_j to the parameter p_j. From this formula, it is clear that one can minimize the effect of parameter fluctuations by tuning system parameters to reduce the magnitude of the parametric impulsive phase response curve. Although (31) has been derived for parameters which fluctuate according to Gaussian white noise statistics, the interpretation of the parametric impulsive phase response curve suggests that this will also be a useful strategy for other types of parameter fluctuations.

The parametric impulsive phase response curve also allows one to understand how an oscillator's period changes due to a (sustained) change in a parameter [34]. In particular, suppose that the period is T for the parameter value p_j, and that at $t = 0$ the parameter $p_j \to p_j + \Delta p$, where Δp is sufficiently small. The change in phase over the time T is given by

$$\Delta\theta \approx \Delta p \int_0^T \left(\sum_{k=1}^{n}\frac{\partial\theta}{\partial x_k}\frac{\partial F_k}{\partial p_j}\right) dt. \quad (32)$$

Using our convention for phase, the change in period associated with this is

$$\Delta T = -\frac{T}{2\pi}\Delta\theta \approx -\Delta p\,\frac{T}{2\pi}\int_0^T\left(\sum_{k=1}^{n}\frac{\partial\theta}{\partial x_k}\frac{\partial F_k}{\partial p_j}\right) dt. \quad (33)$$

Taking the limit $\Delta p \to 0$ gives

$$\frac{\partial T}{\partial p_j} = -\frac{T}{2\pi}\int_0^T\left(\sum_{k=1}^{n}\frac{\partial\theta}{\partial x_k}\frac{\partial F_k}{\partial p_j}\right) dt. \quad (34)$$

Suppose that only one parameter, p_j, fluctuates. If we can tune our system so that

$$\sum_{k=1}^{n}\frac{\partial\theta}{\partial x_k}\frac{\partial F_k}{\partial p_j} = 0 \quad (35)$$

everywhere on the periodic orbit, then these fluctuations have no effect on the phase. This is the generalization of tuning a system with a circular stable periodic orbit to have radial isochrons in order to reduce phase noise associated with a fluctuating parameter which only affects the dynamics in the radial direction. Moreover, if (35) holds, then $\frac{\partial T}{\partial p_j} = 0$, which is related to the conditions in [35] and [36] for reducing the effect of noise on the period of an oscillator.

4. Application to vector fields that do not explicitly depend on phase

Suppose that it is possible to define a phase-like variable ϕ, that the other state variables are constant on the periodic orbit, and that the system dynamics do not depend on ϕ. That is, suppose we can write the state as $\mathbf{x} = (x_1, x_2, \ldots, x_{n-1}, \phi)$, and $\mathbf{F}(\mathbf{x}) = (\mathbf{G}(\mathbf{x}), H(\mathbf{x}))$, where

$$\frac{dx_i}{dt} = G_i(x_1, x_2, \ldots, x_{n-1}), \quad i = 1, \ldots, n-1, \quad (36)$$

$$\frac{d\phi}{dt} = H(x_1, x_2, \ldots, x_{n-1}), \quad (37)$$

and the periodic orbit $\mathbf{x}^\gamma = (x_1^*, x_2^*, \ldots, x_{n-1}^*, \phi)$, where $x_1^*, \ldots, x_{n-1}^*$ are constant, satisfies

$$G_i(x_1^*, x_2^*, \ldots, x_{n-1}^*) = 0, \quad i = 1, \ldots, n-1, \quad (38)$$

$$H(x_1^*, x_2^*, \ldots, x_{n-1}^*) = \omega. \quad (39)$$

It is readily verified that for such a system

$$\nabla_{\mathbf{x}^\gamma}\theta = \begin{pmatrix} -[D\mathbf{G}^T]^{-1}\nabla H \\ 1 \end{pmatrix}, \quad (40)$$

where $\nabla H = \left(\frac{\partial H}{\partial x_1}, \frac{\partial H}{\partial x_2}, \ldots, \frac{\partial H}{\partial x_{n-1}}\right)^T$, and all derivatives are evaluated on the periodic orbit $\mathbf{x}^\gamma$; the above assumptions then imply that $\nabla\theta$ is a constant vector. Eq. (40) can be verified by direct substitution into (5), using the fact that

$$D\mathbf{F}^T = \begin{pmatrix} D\mathbf{G}^T & \nabla H \\ 0_{1\times(n-1)} & 0 \end{pmatrix}.$$

We note that ϕ and θ are the same on the periodic orbit, giving $\frac{\partial\theta}{\partial\phi} = 1$ on the periodic orbit. Moreover, (6) is also satisfied, as verified by using (38) and (39).

We now specialize this to planar systems that can be written in the form

$$\frac{dr}{dt} = G(r), \qquad \frac{d\phi}{dt} = H(r), \tag{41}$$

where r and ϕ are standard polar coordinates in two dimensions, and there is a stable periodic orbit with radius r_{po} (found by solving $G(r_{po}) = 0$), and angular frequency $\omega = H(r_{po})$. Note that these equations can be viewed as a polar coordinate representation of $\lambda - \omega$ systems [37,17]. Then, (40) becomes (cf. [36])

$$\left(\frac{\partial\theta}{\partial r}\bigg|_{\mathbf{x}^\gamma}, \frac{\partial\theta}{\partial\phi}\bigg|_{\mathbf{x}^\gamma}\right) = \left(-\frac{H'(r_{po})}{G'(r_{po})}, 1\right). \tag{42}$$

Transforming to Cartesian coordinates $(x, y) = (r\cos\phi, r\sin\phi)$, and using the fact that $\theta = \phi$ on the periodic orbit, we obtain

$$(Z_1(\theta), Z_2(\theta)) = \left(\frac{\partial\theta}{\partial x}\bigg|_{\mathbf{x}^\gamma}, \frac{\partial\theta}{\partial y}\bigg|_{\mathbf{x}^\gamma}\right)$$
$$= \left(-\frac{H'(r_{po})}{G'(r_{po})}\cos\theta - \frac{\sin\theta}{r_{po}}, -\frac{H'(r_{po})}{G'(r_{po})}\sin\theta + \frac{\cos\theta}{r_{po}}\right), \tag{43}$$

cf. [13,17]. In the case of additive noise, from (27) we then obtain

$$\mathrm{stdev}(T) = \frac{1}{\omega^{3/2}}\left(\pi\left[\left(\frac{H'(r_{po})}{G'(r_{po})}\right)^2 + \frac{1}{r_{po}^2}\right]\sum_{j=1}^{n}\sigma_j^2\right)^{1/2}. \tag{44}$$

For a particular r_{po}, we see that the standard deviation of the period is minimized when $H'(r_{po}) = 0$, which from (42) corresponds to radial isochrons in the neighborhood of the periodic orbit.

4.1. Example: Hopf bifurcation normal form

Consider the vector field

$$\dot{z} = (\alpha + \mathrm{i}\beta)z + (c + \mathrm{i}d)|z|^2 z, \tag{45}$$

where z is complex, and α, β, c, d, f, and g are real. This is the normal form for a Hopf bifurcation [2]. Defining $z = re^{\mathrm{i}\phi} = x + \mathrm{i}y$, we obtain the following alternate forms for (45):

$$\dot{r} = \alpha r + cr^3 \equiv G(r), \tag{46}$$

$$\dot{\phi} = \beta + dr^2 \equiv H(r), \tag{47}$$

and

$$\dot{x} = \alpha x - \beta y + (x^2 + y^2)(cx - dy) \equiv F_1(x, y), \tag{48}$$

$$\dot{y} = \beta x + \alpha y + (x^2 + y^2)(dx + cy) \equiv F_2(x, y). \tag{49}$$

In the following, we will assume that $\alpha > 0$, $c < 0$.

A straightforward dynamical systems analysis shows that there is a fixed point at $z = 0$ with eigenvalues $\alpha \pm \mathrm{i}\beta$. This undergoes a Hopf bifurcation at $\alpha = 0$; since $c < 0$, this is a supercritical Hopf bifurcation. For $\alpha > 0$, this fixed point is unstable, and there is a stable periodic orbit with

$$r_{po} = \sqrt{-\alpha/c}. \tag{50}$$

On the stable periodic orbit, ϕ increases at the constant rate

$$\dot{\phi} = \beta + dr_{po}^2 \equiv \omega = \frac{2\pi}{T}, \tag{51}$$

where T is the period of the periodic orbit.

4.1.1. Additive Gaussian white noise

We modify (48) and (49) to include additive Gaussian white noise:

$$\dot{x} = \alpha x - \beta y + (x^2 + y^2)(cx - dy) + \sigma_1\eta_1(t), \tag{52}$$

$$\dot{y} = \beta x + \alpha y + (x^2 + y^2)(dx + cy) + \sigma_2\eta_2(t), \tag{53}$$

where η_1 and η_2 are real, independent Gaussian white noise random processes with the properties

$$\langle\eta_i(t)\rangle = 0, \qquad \langle\eta_i(t)\eta_j(t')\rangle = \delta_{ij}\delta(t - t'),$$
$$i = 1, 2;\ j = 1, 2. \tag{54}$$

From (44) we obtain

$$\mathrm{stdev}(T) = \frac{1}{\omega^{3/2}}\sqrt{\frac{\pi(c^2 + d^2)(\sigma_1^2 + \sigma_2^2)}{-\alpha c}}. \tag{55}$$

In an application in which an oscillator is designed to have a desired ω (this could be done here by suitably tuning β based on the other parameter values), we see that stdev(T) becomes smaller as d approaches zero, corresponding to $H'(r_{po}) = 0$ and hence radial isochrons in the neighborhood of the periodic orbit. Moreover, stdev(T) becomes smaller as α increases, i.e., as we move further from the bifurcation. This is true even when β is chosen so that ω remains constant. A simple interpretation is that the size of the periodic orbit increases with α, so that noise of the same strength has a relatively smaller effect on the period.

4.1.2. Parameter fluctuations

Now, we consider the case in which the parameter α fluctuates according to

$$\alpha = \alpha_0 + \sigma\eta(t).$$

The parametric impulsive phase response curve associated with fluctuations in α is given by

$$\frac{\partial\theta}{\partial x}\frac{\partial F_1}{\partial\alpha} + \frac{\partial\theta}{\partial y}\frac{\partial F_2}{\partial\alpha} = \frac{\partial\theta}{\partial x}x + \frac{\partial\theta}{\partial y}y, \tag{56}$$

where all expressions are evaluated on the periodic orbit which exists in the absence of noise. Using (43) and $x = \sqrt{-\alpha/c}\cos\theta$, $y = \sqrt{-\alpha/c}\sin\theta$, (31) gives

$$\mathrm{stdev}(T) = \frac{\sqrt{2\pi}}{\omega^{3/2}}\left|\frac{d}{c}\right|\sigma. \tag{57}$$

This expression vanishes when $d = 0$, which has a simple interpretation. When $d = 0$, (47) becomes $\dot{\phi} = \beta$, which is independent of r. Now, from (46), fluctuations in α only affect the r dynamics. Thus, when $d = 0$, we expect fluctuations in α to have no effect on the period of oscillation. Interestingly, here stdev(T) is independent of α_0. Note that $d = 0$ is the condition for radial isochrons.

5. Example: Hindmarsh–Rose neuron

As another example, we consider the Hindmarsh–Rose equations [38], which represent a reduction of the Connor model for crustacean axons [39] and undergo a saddle–node bifurcation of fixed point on a periodic orbit, sometimes called a **SNIPER** bifurcation (Saddle–Node Infinite PERiod) or a **SNIC** bifurcation (Saddle–Node on Invariant Circle) [16]. These equations are given in Appendix A. Here I is treated as the bifurcation parameter, and Gaussian white noise is added to the voltage equation.

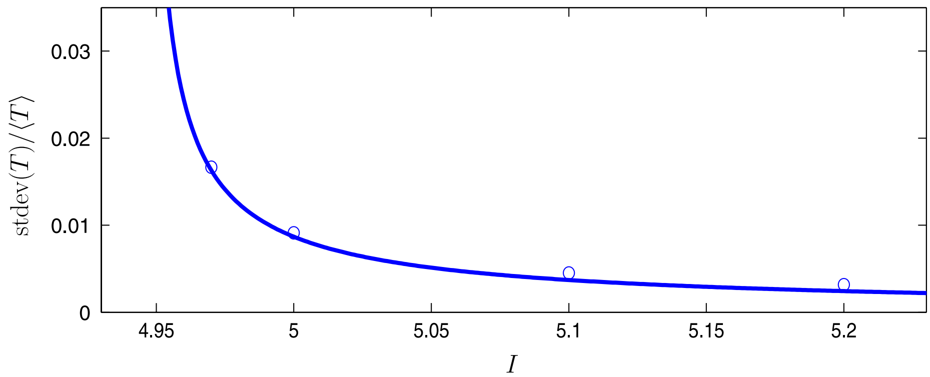

Fig. 1. Relative oscillator precision measured by $\mathrm{stdev}(T)/\langle T\rangle$ for the Hindmarsh–Rose equations as a function of I, as predicted from (61) (shown as a line) and as determined from numerical simulations (shown as circles).

As discussed in [16], cf. [14], the phase response curve corresponding to perturbations in the voltage direction for these equations is

$$\frac{\partial\theta}{\partial V} = Z(\theta) \approx \frac{c_{sn}}{\omega}(1-\cos\theta), \tag{58}$$

where for the parameters under consideration we numerically find that $c_{sn} = 0.0036$. Moreover, we have determined the following fit between ω and I to be a good approximation for the I values of interest:

$$\omega = 0.0799887(I - I_{sn})^{1/2} + 0.0262183(I - I_{sn}), \tag{59}$$

where $I_{sn} = 4.9452$ is the parameter value at which the **SNIPER/SNIC** bifurcation occurs. We note that $\omega \to 0$ as $I \to I_{sn}$, corresponding to the period blowing up to infinity.

From (21), $\langle T\rangle = 2\pi/\omega + \cdots$. Moreover, (27) implies that

$$\mathrm{stdev}(T) = \frac{\sqrt{3\pi}\,c_{sn}}{\omega^{5/2}}\sigma + \cdots. \tag{60}$$

Since the period of the periodic orbit depends on I, as a measure of the relative oscillator precision we consider

$$\frac{\mathrm{stdev}(T)}{\langle T\rangle} = \frac{\sqrt{3\pi}\,c_{sn}}{2\pi\omega^{3/2}}\sigma + \cdots. \tag{61}$$

We see that $\mathrm{stdev}(T)/\langle T\rangle$ decreases as ω increases, that is, as I moves further away from the bifurcation value I_{sn}. This formula is consistent with numerical results, as shown in Fig. 1 for $\sigma^2/2 = 0.0001$.

We note that a similar reduction in magnitude for the phase response curve is expected as one moves away from the bifurcation point for periodic orbits which arise in a supercritical Hopf bifurcation, a homoclinic bifurcation, or a saddle–node bifurcation of periodic orbits (see [16,18]). This suggests the following rule of thumb for improving oscillator precision: operate away from the bifurcation points from which they arise.

6. Improving oscillator precision with impulsive kicks

We have seen above how to tune system parameters to improve oscillator precision. For example, for the Hopf bifurcation normal form with additive noise one would want to tune the system parameters so that the isochrons are radial in the neighborhood of the periodic orbit. Moreover, for a general system with parametric fluctuations, one would want to tune the system parameters to reduce the magnitude of the parametric impulsive phase response curve.

In the following we imagine that we have an oscillator for which such tuning has already been done, or is not possible, and that the oscillator's phase response curve can be measured or calculated. We will show that one can still improve oscillator precision through appropriately-timed impulsive kicks. The key requirements are (i) that the kicks should always change the phase by approximately the same amount, even if they are imprecisely timed, and (ii) that the kicks take the system to a region of phase space where the isochrons are more "diffuse" so that noise will have a smaller effect on the phase. (It is instructive to think of more diffuse isochrons as being associated with phase response curves with smaller magnitude, which lead to greater precision; see (24).) We will illustrate these ideas for the Hopf bifurcation normal form equations in the presence of additive noise, as given in (52) and (53).

6.1. Example: kicking onto the same isochron

We have already seen that Eqs. (52) and (53) with $\alpha > 0$ and $c < 0$ have a stable periodic orbit with $r = r_{po}$, and that tuning $d = 0$ gives optimal oscillator precision. This choice for d corresponds to having radial isochrons, not just in the neighborhood of the periodic orbit, but in the entire basin of attraction of the periodic orbit. Because of this, we can readily determine how an impulsive kick affects the phase for initial conditions either on or off of the periodic orbit.

Specifically, we suppose that when the phase reaches a particular value of θ, an impulsive kick instantaneously takes $x \to x + \Delta x$, corresponding to the phase going from $\theta \to \theta'$; see Fig. 2. Geometrical arguments, cf. [25], can then be used to show that

$$\Delta\theta \equiv \theta' - \theta = -\sin^{-1}\left[\frac{\Delta x \sin\theta}{(r^2 + 2r\Delta x\cos\theta + (\Delta x)^2)^{1/2}}\right]. \tag{62}$$

Expanding this for small Δx, we obtain

$$\Delta\theta = -\frac{\sin\theta}{r}\Delta x + \mathcal{O}((\Delta x)^2). \tag{63}$$

Therefore,

$$Z_1(\theta, r) \equiv \frac{\partial\theta}{\partial x} = \lim_{\Delta x\to 0}\frac{\Delta\theta}{\Delta x} = -\frac{\sin\theta}{r}. \tag{64}$$

Similarly,

$$Z_2(\theta, r) \equiv \frac{\partial\theta}{\partial y} = \frac{\cos\theta}{r}. \tag{65}$$

Note that (64) and (65) characterize the change in phase due to Δx both on ($r = r_{po}$) and off ($r \neq r_{po}$) of the periodic orbit.

Now, suppose that the system is on the periodic orbit with $\theta(0) = 0$, and that at $t = 0$ we apply an impulsive kick taking

$$(x(0^-), y(0^-)) = (r_{po}, 0) \quad\to\quad (x(0^+), y(0^+)) = (r_{po} + \Delta x, 0). \tag{66}$$

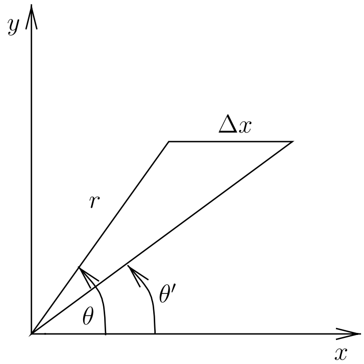

Fig. 2. A perturbation $x \to x + \Delta x$ is applied when the phase reaches θ, kicking the phase to a value θ'.

Because the isochrons are radial, such a kick keeps the system on the same $\theta = 0$ isochron. (More generally, one can find the phase at which a small kick keeps the system on the same isochron by determining where the phase response curve corresponding to the perturbation direction of the kick is zero.) In the absence of noise, and assuming Δx is sufficiently small, the trajectory will return to the (stable) periodic orbit according to the solution of the linearization of (46) about r_{po} with initial condition $r(0^+) = r_{po} + \Delta x$:

$$r(t) \approx r_{po} + \Delta x e^{-2\alpha t}. \tag{67}$$

In the presence of additive noise, we will approximate the precision of the oscillator with once-per-period impulsive kicks at $\theta = 0$ using (27), with generalized phase response curves given by (64) and (65) and $r(t)$ found from (67) with $t = \theta/\omega$, the latter two conditions being approximately true for small noise. That is,

$$\mathrm{stdev}(T) \approx \frac{1}{\omega^{3/2}} \left(\sigma_1^2 \int_0^{2\pi} [Z_1(\Theta, r(\Theta/\omega))]^2 d\Theta + \sigma_2^2 \int_0^{2\pi} [Z_2(\Theta, r(\Theta/\omega))]^2 d\Theta \right)^{1/2}. \tag{68}$$

We expect improvement in oscillator precision when these kicks at $\theta = 0$ have positive Δx, because then we expect to have $|Z_i(\Theta, r(\Theta/\omega))| < |Z_i(\Theta, r_{po})|$; geometrically, the isochrons are more diffuse for larger radius r. (We note that the trajectory approaches the periodic orbit asymptotically, so that for repeated kicks the initial condition assumed above in the calculation of $r(t)$ is only approximate. However, this is not a significant issue if the Floquet multiplier associated with the stability of the periodic orbit gives sufficiently fast approach to the periodic orbit; if this is not the case, the above argument could be adapted to give a more accurate expression for $r(t)$.)

Note that, in general, one can determine where isochrons are more diffuse by recalling that isochrons, being sets of constant asymptotic phase, are tangent to the hyperplanes normal to the gradient of the phase $\nabla\theta$, i.e., the system's (infinitesimal) phase response curve. If it is not possible to find the phase response curve for perturbations in all directions needed to obtain the local approximation to the isochrons, one could determine if positive or negative kicks improve precision through trial and error.

The discussion above assumes that the impulsive kicks always occur precisely at $\theta = 0$. We now demonstrate numerically that improvement in oscillator precision is also possible when the kicks themselves are imprecise, specifically when they occur at θ values drawn from a normal distribution with zero mean and sufficiently small standard deviation σ_k. For definiteness, we take the parameters to be $\alpha = 1, c = -1, \omega = \beta = 2\pi$, and $\sigma_1 = \sigma_2 = \sqrt{2D}$, where $D = 0.00001$. For these parameters, in the absence of impulsive kicks we find that $\langle T \rangle = 1$ and, from (55), $\mathrm{stdev}(T) = 7.12 \times 10^{-4}$, which have been confirmed numerically.

Fig. 3 shows the dynamics with once-per-period impulsive kicks near $\theta = 0$ for $\Delta x = 0.1$ and $\sigma_k = 0.01$. We numerically calculate the standard deviation of crossing times of the surface at $\theta = 3\pi/2$ to be 6.95×10^{-4} (with comparable values for crossings at other θ values), a modest improvement over the case for which there are no impulsive kicks. Such improvement in oscillator precision can be found for ranges of σ_k and Δx values, as shown in Fig. 4. This figure also shows that (68) gives a reasonable approximation for the oscillator precision when $\sigma_k = 0$.

A natural question which arises is how to time the impulsive kicks to be sufficiently close to $\theta = 0$ to be effective. This could be from an escapement mechanism as for a pendulum clock [40]. Alternatively, this could be accomplished through the use of another oscillator as a timer, with its oscillation triggered by a crossing event of the oscillator which is being kicked. We note that one can convert the uncertainty in the kick phases into a time uncertainty according to

$$\frac{\sigma_k}{2\pi} \approx \frac{s_T}{\langle T \rangle},$$

where s_T is the standard deviation of the period of the oscillator which times the kicks. Thus, for this example, to obtain a kick phase uncertainty $\sigma_k = 0.01$, we need another oscillator with period uncertainty $s_T \approx 0.0016$, which is about a factor of two less precise than the oscillator that is being kicked ($\mathrm{stdev}(T) = 7.12 \times 10^{-4}$).

We note that we expect the above results to be robust to small fluctuations in the kick amplitude Δx, because such kicks will still approximately take the system to the same isochron.

6.2. Example: kicking onto different isochrons

We now consider an alternative strategy for improving oscillator precision using impulsive kicks, in which the kicks are designed to take the trajectory onto different isochrons, rather than trying to keep the trajectory on the same isochron as for the previous example. A key insight here is that we want the kicks to advance the phase by (approximately) the same amount even when they are applied at imprecise phases. If this could be done exactly, the imprecise kicks themselves would not cause decreased precision for the oscillator (although they will change the mean period). When such kicks also take the system to a region of phase space where the isochrons are more diffuse, we may get improved oscillator precision.

For definiteness, consider kicks $x \to x + \Delta x$. For small Δx, the change in phase associated with such a kick is

$$\Delta\theta \approx \left.\frac{\partial\theta}{\partial x}\right|_{\mathbf{x}^\gamma} \Delta x. \tag{69}$$

Here $\frac{\partial\theta}{\partial x}|_{\mathbf{x}^\gamma}$ is the phase response curve, and can be thought of as a function of θ, which is the phase at which the kick occurs.

Now suppose that the phase at which kicks occur is imprecise. To avoid having the kicks introduce a new source of oscillator imprecision, we want all of the kicks to advance the phase by (approximately) the same amount. That is, we want to the kicks to occur at or near a value of θ for which $d(\Delta\theta)/d\theta = 0$. Using (69) with $\Delta x \neq 0$, we therefore want to apply kicks when

$$\frac{d}{d\theta}\left(\left.\frac{\partial\theta}{\partial x}\right|_{\mathbf{x}^\gamma} \right) = 0. \tag{70}$$

These ideas are now illustrated for (52) and (53), where for definiteness we take $d = 1$ (giving nonradial isochrons), $\alpha = 1$, $c = -1$, $\omega = \beta = 2\pi$, and $\sigma_1 = \sigma_2 = \sqrt{2D}$, where $D = 0.00001$.

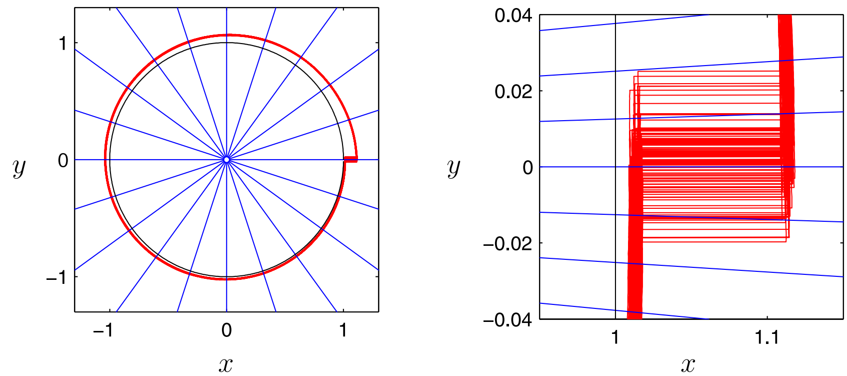

Fig. 3. Dynamics of the Hopf bifurcation normal form with additive noise and parameters given in the main text, with once-per-period impulsive kicks at $\theta \approx 0$ with $\Delta x = 0.1$ and $\sigma_k = 0.01$. In the left panel, the periodic orbit in the absence of impulsive kicks is also shown as a circle, and isochrons spaced in phase by $\pi/10$ are shown as radial lines. In the zoomed-in right panel, the periodic orbit in the absence of kicks looks like a straight vertical line at $x = 1$, and isochrons spaced in phase by $\pi/250$ are shown.

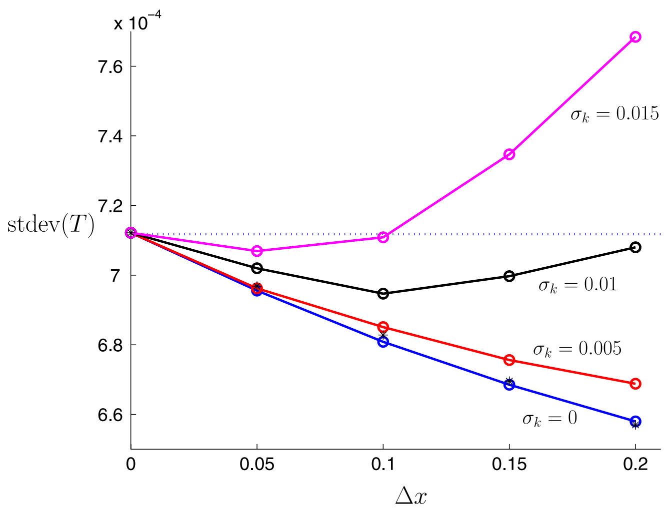

Fig. 4. Oscillator precision with once-per-period impulsive kicks of size Δx and kick precision σ_k for the Hopf bifurcation normal form with additive noise and parameters as given in the main text. The $*$'s indicate predictions from (68) for $\sigma_k = 0$. The horizontal dashed line shows the oscillator precision in the absence of kicks.

For these parameters, in the absence of kicks our numerical calculations find the mean period to be $\langle T\rangle = 0.862$, the standard deviation of the period to be $\text{stdev}(T) = 7.04 \times 10^{-4}$, and the relative precision to be $\text{stdev}(T)/\langle T\rangle = 8.17 \times 10^{-4}$. Moreover, (43) implies that

$$\left.\frac{\partial \theta}{\partial x}\right|_{\mathbf{x}^\gamma} = \cos\theta - \sin\theta. \tag{71}$$

Then, (70) implies that a good phase to kick the oscillators at is $\theta_k = -\pi/4$.

Fig. 5 shows the dynamics with once-per-period impulsive kicks with mean $\theta = -\pi/4$ for $\Delta x = 0.075$ and $\sigma_k = 0.05$. The isochrons in this figure are exact, according to (73) from Appendix B. We numerically calculate the mean of the crossing times to be $\langle T\rangle = 0.848$, the standard deviation to be $\text{stdev}(T) = 6.83 \times 10^{-4}$, and the relative precision to be $\langle T\rangle/(\text{stdev}(T)) = 8.05 \times 10^{-4}$. Thus, the relative precision is improved compared with the case of no kicks. Improvement in relative oscillator precision can be found for various values of σ_k and Δx values, as shown in Fig. 6. We emphasize that the timing of the kicks is important: if once-per-period kicks occur with mean $\theta = 0$ for $\Delta x = 0.075$ and $\sigma_k = 0.05$, the mean of the crossing times is $\langle T\rangle = 0.853$, the standard deviation is $\text{stdev}(T) = 8.37 \times 10^{-4}$, and the relative precision is $\langle T\rangle/(\text{stdev}(T)) = 9.81 \times 10^{-4}$; this is worse relative precision than we would obtain by not kicking the oscillator at all.

Note that we can predict $\langle T\rangle$ for $\sigma_k = 0$ by recognizing that the change in phase due to the kick causes a change in the expected period:

$$\frac{\langle T\rangle_{\text{with kick}}}{\langle T\rangle_{\text{no kick}}} \approx \frac{2\pi - \Delta\theta}{2\pi} \approx \frac{2\pi - \left.\frac{\partial\theta}{\partial x}\right|_{\mathbf{x}^\gamma}\Delta x}{2\pi}.$$

For kicks at $\theta = -\pi/4$, $\left.\frac{\partial\theta}{\partial x}\right|_{\mathbf{x}^\gamma} = \sqrt{2}$, so

$$\langle T\rangle_{\text{with kick}} \approx T\left(1 - \frac{\sqrt{2}\Delta x}{2\pi}\right) = \frac{2\pi}{\omega}\left(1 - \frac{\sqrt{2}\Delta x}{2\pi}\right). \tag{72}$$

Using this formula with (51) gives the $*$'s in the top panel of Fig. 6.

Although the strategy presented here can improve oscillator precision, and is robust to small fluctuations in the phase at which the kicks occur, the oscillator precision will be reduced by fluctuations in the kick amplitude Δx, a consequence of the geometry of the isochrons (see Fig. 5).

7. Conclusion

We have considered how the period of an oscillator is affected by noise, with special attention given to the cases of additive noise and parameter fluctuations. Our treatment was based upon the concepts of isochrons, which provide the natural coordinate system for calculating the precision of the period for an oscillator, and phase response curves, which can be used to understand the geometry of isochrons in the neighborhood of the periodic orbit. This included the derivation of the effect of noise on an oscillator's period, with formulas for the mean and standard deviation of the period to leading order in the strength of the noise. Several examples were considered in detail which illustrate the use and validity of the theory, and suggest how to optimize an oscillator's precision by tuning system parameters. For more general oscillators, one may be able to use center manifold reduction and normal form transformations to express the system in new coordinates (for example, the Hopf bifurcation normal form) for which the phase response curve can be calculated analytically in terms of its parameters; one can then transform back to the original coordinates to obtain an expression for the phase response curve that can be used in the formulas for the standard deviation of the oscillator's period, ultimately allowing optimization of the oscillator's precision with respect to parameters.

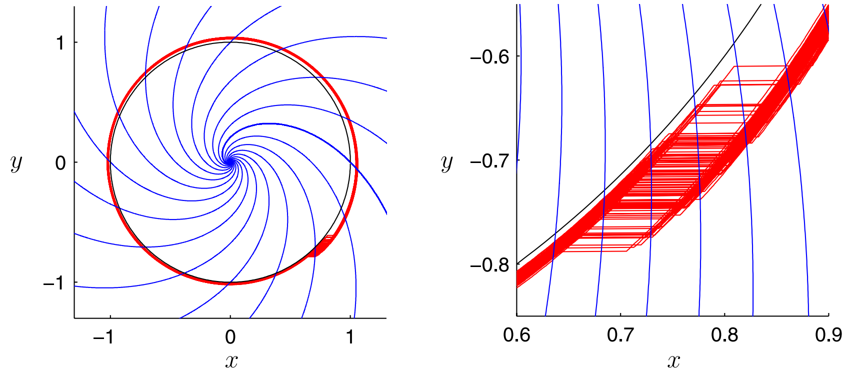

Fig. 5. Dynamics of the Hopf bifurcation normal form with additive noise and parameters given in the main text, with once-per-period impulsive kicks at $\theta \approx -\pi/4$, with $\Delta x = 0.075$ and $\sigma_k = 0.05$. In the left panel, the periodic orbit in the absence of impulsive kicks is also shown as a circle, and isochrons spaced in phase by $\pi/16$ are shown. In the zoomed-in right panel, isochrons spaced in phase by $\pi/250$ are shown.

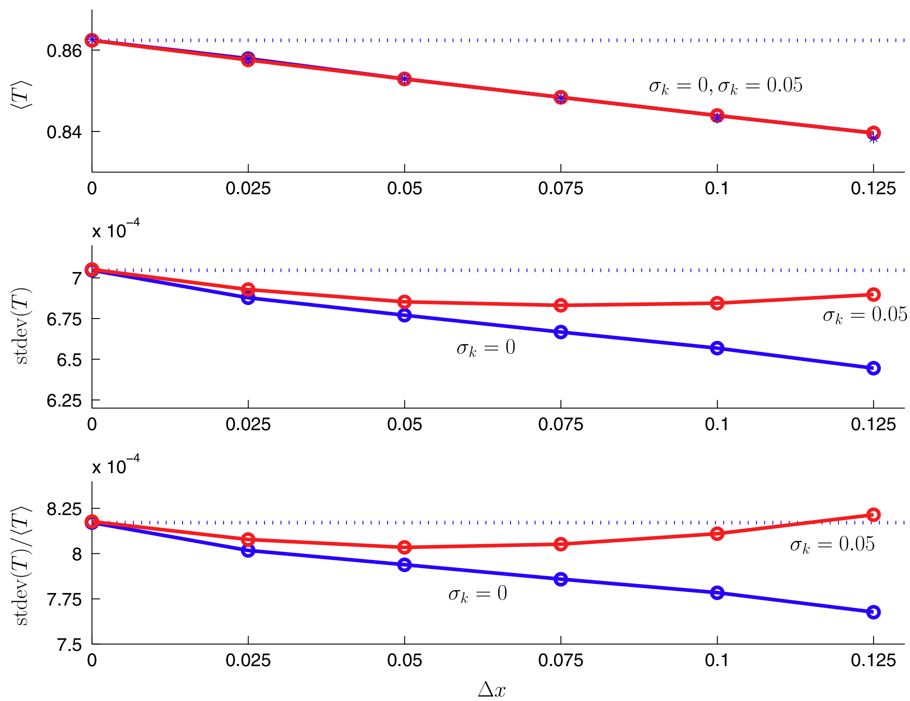

Fig. 6. Oscillator properties with once-per-period impulsive kicks of size Δx and kick precision σ_k for the Hopf bifurcation normal form with additive noise and parameters given in the main text. (Top) The mean period $\langle T\rangle$, with $*$'s showing predicted values from (72). (Middle) The standard deviation of the period. (Bottom) Relative oscillator precision measured by $\mathrm{stdev}(T)/\langle T\rangle$. In all panels, the horizontal dashed line shows the respective value in the absence of kicks.

Moreover, we proposed a method for improving oscillator precision through the use of appropriately-timed impulsive kicks designed to always change the phase by approximately the same amount, even when the kick timing is imprecise, and also kick into a region of phase space for which isochrons are more diffuse so that noise has a smaller effect on the phase. This approach was illustrated for the Hopf bifurcation normal form with additive noise.

There are several ways in which these results in this paper could be improved. Most notably, the calculations are only valid to leading order in the noise strength. By keeping the appropriate $\mathcal{O}(\sigma^2)$ terms in the phase reduction with noise [32], and keeping the appropriate Taylor expansions to higher order, one could obtain more accurate predictions for $\langle T\rangle$ and $\mathrm{stdev}(T)$, cf. [27]. Moreover, more accurate approximations for the isochrons could be used (e.g. [22,21]) to better capture the effect of larger noise and impulsive kicks. Results can also be obtained and analyzed for other types of noise [41,32,33]. Finally, we mention the interesting results in [27] which suggest that oscillator precision can be improved through coupling to other oscillators.

Acknowledgments

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Appendix A. The Hindmarsh–Rose equations

The Hindmarsh–Rose equations that we study are given by:

$$\dot{V} = [I - \bar{g}_{Na} m_\infty(V)^3(-3(q - Bb_\infty(V)) + 0.85)(V - V_{Na}) - \bar{g}_K q(V - V_K) - \bar{g}_L(V - V_L)]/C + \sigma\eta(t),$$

$$\dot{q} = (q_\infty(V) - q)/\tau_q(V),$$
$$\langle \eta(t) \rangle = 0, \qquad \langle \eta(t)\eta(t') \rangle = \delta(t - t'),$$
$$q_\infty(V) = n_\infty(V)^4 + Bb_\infty(V),$$
$$b_\infty(V) = (1/(1 + \exp(\gamma_b(V + 53.3))))^4,$$
$$m_\infty(V) = \alpha_m(V)/(\alpha_m(V) + \beta_m(V)),$$
$$n_\infty(V) = \alpha_n(V)/(\alpha_n(V) + \beta_n(V)),$$
$$\tau_q(V) = (\tau_b(V) + \tau_n(V))/2,$$
$$\tau_n(V) = T_n/(\alpha_n(V) + \beta_n(V)),$$
$$\tau_b(V) = T_b(1.24 + 2.678/(1 + \exp((V + 50)/16.027))),$$
$$\alpha_n(V) = 0.01(V + 45.7)/(1 - \exp(-(V + 45.7)/10)),$$
$$\alpha_m(V) = 0.1(V + 29.7)/(1 - \exp(-(V + 29.7)/10)),$$
$$\beta_n(V) = 0.125 \exp(-(V + 55.7)/80),$$
$$\beta_m(V) = 4 \exp(-(V + 54.7)/18).$$
$$V_{Na} = 55 \text{ mV}, \qquad V_K = -72 \text{ mV}, \qquad V_L = -17 \text{ mV},$$
$$\bar{g}_{Na} = 120 \text{ mS/cm}^2,$$
$$\bar{g}_K = 20 \text{ mS/cm}^2, \qquad \bar{g}_L = 0.3 \text{ mS/cm}^2, \qquad g_A = 47.7 \text{ mS/cm}^2,$$
$$C = 1\ \mu F/\text{cm}^2, \qquad \gamma_b = 0.069 \text{ mV}^{-1},$$
$$T_b = 1 \text{ ms}, \qquad T_n = 0.52 \text{ ms}, \qquad B = 1.26.$$

Appendix B. Isochrons for Hopf bifurcation normal form

For the Hopf bifurcation normal form, the isochrons can be found analytically throughout the entire phase space, as follows. First, (46) can be solved to give

$$r(t) = \frac{\sqrt{\alpha} r_0}{\sqrt{\alpha + c r_0^2}} \frac{e^{\alpha t}}{\sqrt{1 - \frac{c r_0^2}{\alpha + c r_0^2} e^{2\alpha t}}},$$

where $r_0 \equiv r(0)$. Using this, (47) can be solved to give

$$\phi(t) = \phi_0 + \frac{d}{2c} \log \alpha + \beta t - \frac{d}{2c} \log\left(\alpha - c(e^{2\alpha t} - 1) r_0^2\right),$$

where $\phi_0 \equiv \phi(0)$. In the limit as $t \to \infty$, the exponential in the last log term will dominate, so that

$$\begin{aligned}
\phi &\approx \phi_0 + \frac{d}{2c} \log \alpha + \beta t - \frac{d}{2c} \log(-c e^{2\alpha t} r_0^2) \\
&= \phi_0 + \frac{d}{2c} \log \alpha + \beta t - \frac{d}{2c} \log(-c r_0^2) - \frac{d\alpha}{c} t \\
&= \phi_0 + \frac{d}{2c} \log \alpha - \frac{d}{2c} \log(-c r_0^2) + \omega t,
\end{aligned}$$

where the second equation follows from the properties of logarithms, and the third equation follows from the definition of ω in (51). Therefore, all initial conditions that lie on the curve defined by the equation

$$\phi + \frac{d}{2c} \log \alpha - \frac{d}{2c} \log(-c r^2) = \text{constant} \tag{73}$$

asymptotically approach the periodic orbit with the same phase; thus, (73) gives the formula for isochrons, with different isochrons corresponding to different values for the constant.

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