Eckhaus-Benjamin-Feir instability in systems with temporal modulation

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(Received 18 June 1996)

A general method for computing the generalized Eckhaus boundary for the complex Ginzburg-Landau equation with a time-periodic control parameter is given. The case in which the control parameter undergoes small sinusoidal oscillations about a constant value is worked out explicitly, and the frequency dependence of the result is discussed. [S1063-651X(96)14611-0]

PACS number(s): 47.20.Ky, 47.54.+r

I. INTRODUCTION

The Ginzburg-Landau equation

$$A_t = \mu A + A_{xx} - |A|^2 A$$

first arose in the context of superconductivity [1], and has since been derived [2–6] and tested experimentally [7–11] as an amplitude equation describing the slow (in both space and time) evolution of one-dimensional patterns near onset of a steady-state pattern-forming instability [12]. In this equation μ is the control parameter and the complex amplitude $A(x,t)$ is related, for example, to a stream function ψ by $\psi = \epsilon \mathrm{Re}(Ae^{ik_c x/\epsilon}) + O(\epsilon^2)$. Here k_c is the wave number of the pattern at onset and ϵ^2 is a small parameter denoting distance from the instability threshold. Near onset of a long wavelength oscillatory instability [12] the equation generalizes to the complex Ginzburg-Landau equation

$$A_t = \mu A + (1+i\alpha)A_{xx} - (1+i\beta)|A|^2 A,$$

where α and β are real constants that describe linear and nonlinear dispersive effects. This equation also describes the evolution of the amplitude of a traveling wave with finite wave number k_c in a reference frame moving at the group velocity of the wave.

In this paper we are interested in the stability properties of *nonlinear* plane wave solutions of both equations. Such solutions take the form $A = K(k)e^{ikx+i\Omega t}$ and represent patterns with wave number $k_c + \epsilon k$, i.e., patterns with a slightly different wavelength from that selected at onset. The stability of such solutions has been studied for both the real [13–15] and complex [16–18] Ginzburg-Landau equations; in the Hamiltonian case the corresponding analysis goes back to Benjamin and Feir [19,20]. If $\mu < 0$ the only solution is the trivial (i.e., spatially uniform) state $A = 0$. If $\mu > 0$ the trivial state is linearly unstable to solutions of the form $A = K(k)e^{ikx+i\Omega t}$ whose wave number satisfies $k^2 \leqslant \mu$. However, if $1+\alpha\beta > 0$, these solutions are themselves unstable in the limit of infinite wavelength perturbations unless $[(3+\alpha\beta+2\beta^2)/(1+\alpha\beta)]k^2 \leqslant \mu$, with equality defining what we will call the generalized Eckhaus boundary. If $1+\alpha\beta < 0$ all solutions with $k^2 \leqslant \mu$ are unstable. In particular, if we fix α and continuously vary β such that $1+\alpha\beta$ changes from a positive to a negative value, the parabola in (k,μ) space describing the generalized Eckhaus boundary shrinks in width, vanishing when $1+\alpha\beta = 0$. It should be emphasized that this notion only pertains to the stability properties of plane waves with respect to long wavelength perturbations. For the real Ginzburg-Landau equation these are in fact the most dangerous perturbations; in the complex case the long wavelength perturbations need not be the most dangerous ones, and the above condition determines the stability properties of plane waves only in certain regions in the (α,β) plane [17]. The Eckhaus instability is responsible for (partial) wavelength selection in one-dimensional systems, and can lead to spatio-temporal complexity. An example of particular interest arises when spatial ramps are used to select a wave number outside of the Eckhaus-stable band; in this case the system is forced to undergo repeated phase slips which may occur periodically or chaotically as a result of repeated triggering of the Eckhaus instability [21,22]. The instability can also be triggered, in a less organized manner, by subjecting the system to spatially varying noise. In [23] the effect of additive stochastic fluctuations on the Eckhaus boundary for the Swift-Hohenberg equation was examined; it was found that such fluctuations (which are not to be interpreted as fluctuations in the control parameter) cause the band of stable wave numbers to be reduced. The effect of stochastic fluctuations in the control parameter on the primary instability has also been considered [24] with a view to determining the shift in the onset of the instability. However, fluctuations in μ have another effect as well. They change the instantaneous width of the band of growing wave numbers and so move a certain range of wave numbers repeatedly into and out of the Eckhaus stable region. The cumulative effect of such oscillations is to shift the Eckhaus stability boundary, and its description is the subject of this paper.

We focus on the effects of temporally periodic modulation of the control parameter on the Eckhaus boundary. As is well known, temporal modulation may shift the threshold for the onset of the primary instability [25–27], can lead to pattern selection [28–30], and, in the presence of noise, can affect transitions between attractors [31,32]. In the present paper, we show that such modulation also affects the generalized Eckhaus boundary, and obtain an analytic expression for the boundary for small amplitude sinusoidal modulation. In the special case of the real Ginzburg-Landau equation with a sinusoidally modulated control parameter it is found that the band of stable wave vectors is always reduced, with lower modulation frequencies giving greater reduction. For the complex Ginzburg-Landau equation the details are sensitive to the values of α, β, and the modulation frequency, as elaborated further below.

II. THE ECKHAUS-BENJAMIN-FEIR INSTABILITY WITH PERIODIC MODULATION

In this section we describe the technique we use to study the effect of periodic modulation of the control parameter on the Eckhaus-Benjamin-Feir instability. We generalize the complex Ginzburg-Landau equation to

$$A_t = \mu(t)A + (1+i\alpha)A_{xx} - (1+i\beta)|A|^2 A, \tag{1}$$

where $\mu(t)$ is assumed to be differentiable and $\mu(t+T)=\mu(t)$. Without loss of generality $\mu(t)$ can be taken to be real. Also, for simplicity, the effect of the sidewalls is ignored, and it is assumed that the only effect of modulation is on the control parameter. As already mentioned, a wide class of pattern-forming systems can be reduced to Eq. (1) near the onset of instability. It is important to observe that the time scale for the variation of the control parameter is taken to be the *same* as that for the growth and equilibration of the pattern-forming instability. In terms of the parameters of the physical system this implies an appropriately slow modulation of the control parameter. For solutions of the form

$$A(x,t) = R(t)\exp\left[ikx + i\int \Omega(t')dt'\right], \tag{2}$$

where $R(t)$ is real, we obtain

$$R_t = \mu(t)R - k^2 R - R^3,$$

$$\Omega = -\alpha k^2 - \beta R^2.$$

Linearization about the trivial solution $(A=R=0)$ gives

$$R(t) \sim \exp\left[\int^t [\mu(t') - k^2]dt'\right],$$

and since the integrand is periodic, the trivial solution is linearly unstable to solutions of the form (2) if the average value of $\mu(t)$ is greater than k^2:

$$\frac{1}{T}\int_0^T \mu(t)dt > k^2. \tag{3}$$

This condition generalizes the usual instability condition $\mu > k^2$ to the time-dependent case, and will be assumed in what follows.

The stability of solutions of the form (2) is investigated by considering the solution

$$A(x,t) = \exp\left[ikx + i\int \Omega(t')dt'\right][R(t) + a(x,t)],$$

where $a(x,t)$ is an infinitesimal perturbation of the form

$$a(x,t) = a^+(t)e^{imx} + \overline{a^-}(t)e^{-imx}.$$

The linearized equations for the amplitudes a^+ and a^- are

$$a_t^+ = [\mu(t) - i\Omega - (k+m)^2(1+i\alpha) - 2(1+i\beta)p(t)]a^+ - (1+i\beta)p(t)a^-, \tag{4}$$

$$a_t^- = [\mu(t) + i\Omega - (k-m)^2(1-i\alpha) - 2(1-i\beta)p(t)]a^- - (1-i\beta)p(t)a^+, \tag{5}$$

where $p(t) \equiv (R(t))^2$ is real. Note that if $\alpha \neq 0$ and $\beta \neq 0$, then taking the complex conjugates of these equations gives two more independent equations. However, the equations for a^+ and a^- are uncoupled from those for $\overline{a^+}$ and $\overline{a^-}$.

An important property of $p(t)$ is that in the long-time limit it becomes periodic with the same period as $\mu(t)$. To prove this we note that

$$\frac{dp}{dt} = 2RR_t = 2[\mu(t) - k^2]p - 2p^2,$$

or, equivalently,

$$\frac{dq}{dt} + 2[\mu(t) - k^2]q = 2, \tag{6}$$

where $q(t) \equiv 1/p(t)$. Thus

$$q(t) = q_h(t) + q_p(t),$$

where

$$q_h(t) \sim \exp\left[-2\int^t [\mu(t') - k^2]dt'\right]$$

and

$$q_p(t) = 2\exp\left[-2\int^t [\mu(t') - k^2]dt'\right] \times \int^t \exp\left[2\int^{t'} [\mu(t'') - k^2]dt''\right]dt'.$$

The particular solution q_p satisfies Eq. (6) at times t and $t+T$, so that

$$\frac{dq_p}{dt}(t+T) + 2[\mu(t+T) - k^2]q_p(t+T) = 2$$

and

$$\frac{dq_p}{dt}(t) + 2[\mu(t) - k^2]q_p(t) = 2.$$

Using the periodicity of $\mu(t)$, these equations yield

$$\frac{d}{dt}[q_p(t+T) - q_p(t)] = -2[\mu(t) - k^2][q_p(t+T) - q_p(t)]$$

and hence

$$q_p(t+T)-q_p(t)\sim\exp\left[-2\int^t[\mu(t')-k^2]dt'\right].$$

It follows, from condition (3), that as $t\to\infty$, $q(t)\to q_p(t)$, and that $q_p(t)$ is T-periodic in this limit, and correspondingly that $p(t)$ is also T-periodic.

Equations (4) and (5) can be simplified by setting

$$a^{\pm}(t)=\exp\left[\int^t[2\mu(t')-2k^2-3p(t')-m^2-2i\alpha km]dt'\right]\times c^{\pm}(t), \tag{7}$$

whence

$$c_{tt}^{\pm}+\left[\frac{d\mu}{dt}-[\mu(t)-k^2]^2\mp 4mk[\mu(t)-k^2-p(t)]-4m^2k^2+\alpha^2m^4+2\alpha\beta m^2p(t)\right]c^{\pm}-2im\{\pm\alpha m[\mu(t)-k^2-p(t)]+2\alpha km^2+2\beta kp(t)\}c^{\pm}=0. \tag{8}$$

Since both μ and p are T-periodic in the long-time limit, we see that, in this limit, Eq. (8) is a differential equation of Mathieu type, with (complex) periodic coefficients. Only when $\alpha=\beta=0$ is the equation for $c^{\pm}$ real.

We now restrict our attention to the case where the control parameter is undergoing small oscillations about a constant value. Since the time scale for the evolution of the instability a distance $O(m^2)$ from the generalized Eckhaus boundary scales as $O(m^{-4})$, where m is the perturbation wave number, we anticipate that nontrivial effects will arise in this region when the modulation effects occur precisely on this time scale. We introduce a small parameter δ measuring the amplitude of the modulation of the control parameter and consider perturbation wave numbers m of order δ,

$$m=\delta\tilde{m}, \tag{9}$$

where $\tilde{m}$ is $O(1)$. Here δ is defined in terms of the expansion

$$\mu(t)=\tilde{\mu}_0+\delta\tilde{\mu}_1(t)+\delta^2\tilde{\mu}_2+\cdots, \tag{10}$$

where $\tilde{\mu}_0,\tilde{\mu}_1(t),\tilde{\mu}_2,\ldots$ are $O(1)$ and

$$\tilde{\mu}_0=\frac{(3+\alpha\beta+2\beta^2)}{1+\alpha\beta}k^2. \tag{11}$$

Thus the time-independent quantity $\eta\equiv\tilde{\mu}_2$ indicates the distance from the generalized Eckhaus boundary (11). In the following we will vary η in order to explore the vicinity of this boundary, and in particular to search for the values $\eta_{\rm EBF}$ at which the growth rate of the instability vanishes. Recall that when $\tilde{\mu}_1=0$ this occurs along the Eckhaus boundary. Thus $\eta_{\rm EBF}=0$ when $\tilde{\mu}_1=0$. When modulation is present we expect that the Eckhaus boundary will be shifted relative to Eq. (11), and consequently that $\eta_{\rm EBF}\neq 0$. We now explain how we calculate $\eta_{\rm EBF}$.

For reasons already mentioned, the required calculation needs to be carried to fourth order in the parameter δ. We first calculate the quantity $p(t)$ in powers of δ,

$$p(t)=\tilde{p}_0+\delta\tilde{p}_1(t)+\delta^2\tilde{p}_2(t)+\cdots, \tag{12}$$

where $\tilde{p}_0=\tilde{\mu}_0-k^2$ and $\tilde{p}_1(t),\tilde{p}_2(t),\ldots$ are $O(1)$. Explicit expressions for the $\tilde{p}_j$, $j=1,\ldots,4$, in the long-time limit can be found in Appendix A. Substituting Eqs. (9), (10), and (12) into Eq. (8) and collecting powers of δ, we obtain

$$c_{tt}^{\pm}+\left[\sum_{j=0}^{\infty}\delta^j f_j^{\pm}(t)\right]c^{\pm}=0, \tag{13}$$

where $f_0^{\pm}(t),f_1^{\pm}(t),\ldots$ are $O(1)$. We solve this equation using the method of multiple scales. Thus, we let

$$c^{\pm}(t)=\sum_{i=0}^{\infty}\delta^i c_i^{\pm}(T_0,T_1,T_2,\ldots),$$

where $T_n\equiv\delta^n t$. Equation (13) then yields a hierarchy of equations for the $c_i^{\pm}$ (see Appendix B) with solution of the form

$$c_j^{\pm}(T_0,T_1,T_2,\ldots)=\exp\left[\sum_{i=0}^{\infty}\hat{s}_i^{\pm}T_i\right]y_j^{\pm}(T_0). \tag{14}$$

Here the $y_i^{\pm}$ are T-periodic in the fast variable T_0 and the $\hat{s}_i^{\pm}$ are (Floquet) exponents which remain to be determined. Note that there are in fact two Floquet exponents for both c^+ and c^-, but for the stability analysis it suffices to restrict attention to the one with the greatest real part. To determine the growth rate of the perturbations $a^{\pm}(t)$ we note that

$$\int^t[2\mu(t')-2k^2-3p(t')-m^2-2i\alpha km]dt'=u(t)+\sum_{i=0}^{\infty}\tilde{s}_iT_i, \tag{15}$$

where $u(t+T)=u(t)$, and hence using Eq. (7) that

$$a^{\pm}(t)=\exp\left[\sum_{i=0}^{\infty}\delta^i(\tilde{s}_i+\hat{s}_i^{\pm})t\right]v^{\pm}(t),$$

where

$$v^{\pm}(t)=\exp[u(t)]\sum_{j=0}^{\infty}\delta^j y_j^{\pm}(t)$$

is T-periodic. Thus, the (complex) growth rate of $a^{\pm}(t)$ is

$$s^{\pm}=\sum_{i=0}^{\infty}\delta^i(\tilde{s}_i+\hat{s}_i^{\pm}). \tag{16}$$

The growth rates $\tilde{s}_i$ and $\hat{s}_i^{\pm}$, and hence $s^{\pm}$, depend on $\eta,\tilde{m},k$, and the nature of the modulation, such as its frequency. Once explicit expressions for $s^{\pm}$ are known, the shift in the generalized Eckhaus boundary, $\eta_{\rm EBF}$, follows from the equation $\mathrm{Re}(s^{\pm})=0$. Since $s^{\pm}$ is determined perturbatively it suffices to calculate $\mathrm{Re}(s^{\pm})$ at the lowest nonvanishing order. Note that if $\alpha\neq 0$, $\beta\neq 0$, a similar analysis starting with the complex conjugates of Eqs. (4) and (5) shows that the growth rates of $\overline{a^{\pm}}$ are $\overline{s^{\pm}}$, i.e., the above calculation yields the complete stability information required.

III. AN EXAMPLE: A SMALL AMPLITUDE SINUSOIDAL PERTURBATION TO A CONSTANT CONTROL PARAMETER

As an example we consider the case of small sinusoidal oscillations about $\mu=\tilde{\mu}_0+\delta^2\eta$ and take $\tilde{\mu}_1=\sin\omega t$. Thus the amplitude of the oscillations is δ, and this amplitude is larger than the adopted $O(\delta^2)$ distance from the generalized Eckhaus boundary $\mu=\tilde{\mu}_0$. We restrict attention to the case $1+\alpha\beta>0$ for which this boundary is present. From Appendix A, we obtain

$$\tilde{p}_0=\frac{2(1+\beta^2)k^2}{1+\alpha\beta},$$

$$\tilde{p}_1(t)=\frac{4(1+\beta^2)k^2}{16(1+\beta^2)^2k^4+(1+\alpha\beta)^2\omega^2}[4(1+\beta^2)k^2\sin\omega t-(1+\alpha\beta)\omega\,\cos\omega t],$$

with similar expressions for $\tilde{p}_2(t)$, $\tilde{p}_3(t)$, and $\tilde{p}_4(t)$. In particular $\tilde{p}_2(t)$ equals the constant term η plus terms proportional to $\sin(2\omega t)$ and $\cos(2\omega t)$. Moreover, $\tilde{p}_3(t)$ and $\tilde{p}_4(t)$ only contain terms proportional to $\sin(n\omega t)$ and $\cos(n\omega t)$, with $n=1$ and $n=3$ for $\tilde{p}_3$ and $n=2$ and $n=4$ for $\tilde{p}_4$. We now substitute the expansions for $\mu(t)$ and $p(t)$, along with $m=\delta\tilde{m}$, into Eq. (8) to obtain the $f_i^{\pm}$. For brevity, we only list $f_0^{\pm}$ and $f_1^{\pm}$:

$$f_0^+=f_0^-=\frac{-4(1+\beta^2)^2k^4}{(1+\alpha\beta)^2}\equiv-\lambda^2(\mathrm{say}),$$

$$f_1^+(t)=f_1^-(t)=\frac{-8i\beta(1+\beta^2)k^3\tilde{m}+\omega(1+\alpha\beta)\cos\omega t-4(1+\beta^2)k^2\sin\omega t}{1+\alpha\beta}.$$

In terms of the notation $D_i\equiv\partial/\partial T_i$ the equation for $c_0^{\pm}$ is thus (see Appendix B)

$$D_0^2c_0^{\pm}-\lambda^2c_0^{\pm}=0,$$

so that

$$c_0^{\pm}=A^{\pm}(T_1,T_2,\dots)e^{\lambda T_0}+B^{\pm}(T_1,T_2,\dots)e^{-\lambda T_0}.$$

If we choose $\lambda>0$, then in the long-time limit the second term becomes vanishingly small relative to the first term and so is of no interest for the stability analysis; without loss of generality we therefore take $B^{\pm}=0$. Thus, according to the notation in Eq. (14), $\hat{s}_0^+=\hat{s}_0^-=\lambda$. At next order we obtain the inhomogeneous problem

$$D_0^2c_1^{\pm}-\lambda^2c_1^{\pm}=-f_1^{\pm}(t)A^{\pm}(T_1,T_2,\dots)e^{\lambda T_0}-2\lambda e^{\lambda T_0}D_1A^{\pm}.$$

At each order the solvability condition is that the $c_i^{\pm}$ do not contain terms proportional to $T_0e^{\lambda T_0}$. Such terms would come from inhomogeneous terms proportional to $e^{\lambda T_0}$, but not from terms proportional to $\sin(\omega T_0)e^{\lambda T_0}$ and $\cos(\omega T_0)e^{\lambda T_0}$. The solvability condition at this order is thus

$$D_1A^{\pm}=2i\beta\tilde{m}kA^{\pm},$$

so that $\hat{s}_1^+=\hat{s}_1^-=2i\beta\tilde{m}k$. To go to next order, we need to solve for $c_1^{\pm}(t)$ subject to the solvability condition. Ignoring the solution to the homogeneous equation, we find that

$$c_1^{\pm}(t)=\frac{-2\gamma_2\lambda-\gamma_1\omega}{4\lambda^2\omega+\omega^3}e^{\lambda T_0}\cos(\omega T_0)+\frac{2\gamma_1\lambda-\gamma_2\omega}{4\lambda^2\omega+\omega^3}e^{\lambda T_0}\sin(\omega T_0),$$

where γ_1 and γ_2 are both proportional to $A^{\pm}(T_1,T_2,\dots)$. Hence the growth rate of $c_1^{\pm}$ is the same as the growth rate of $c_0^{\pm}$. In fact, to all orders the c_i have the same growth rate, as indicated in Eq. (14). The quantities $\hat{s}_2^{\pm}$, $\hat{s}_3^{\pm}$, and $\hat{s}_4^{\pm}$ are calculated in a similar way, and complete the calculation of the dominant Floquet multipliers of $c^{\pm}$ to the required order. We omit the details of this calculation.

In order to compute the growth rate of the perturbations of interest, namely $a^{\pm}$, we next calculate the corresponding multipliers from the prefactor in Eq. (7). From the integral in Eq. (15), we obtain

$$\tilde{s}_0=-\lambda,$$

$$\tilde{s}_1=-2i\alpha\tilde{m}k,$$

$$\tilde{s}_2=-(\eta+\tilde{m}^2),$$

$$\tilde{s}_3=\tilde{s}_4=0.$$

The final growth rates for $a^{\pm}(t)$ follow from Eq. (16), and are

$$s^+=s^-=\delta\sigma_1+\delta^3\sigma_3+\delta^4\sigma_4+O(\delta^5), \tag{17}$$

where

$$\sigma_1=2i(\beta-\alpha)\tilde{m}k,$$

$$\sigma_3=-i\tilde{m}\left\{\frac{(1+\alpha\beta)(\beta-\alpha)\tilde{m}^2}{(1+\beta^2)k}+\frac{2\beta(1+\alpha\beta)^2k}{[16(1+\beta^2)^2k^4+(1+\alpha\beta)^2\omega^2]}+\frac{\beta k}{\omega^2}\right\},$$

and

$$\sigma_4=\rho_1\tilde{m}^2+\rho_2\tilde{m}^4,$$

with

$$\rho_1=(1+\alpha\beta)\left\{\frac{32(1+\alpha\beta)^2(1+\beta^2)k^4}{[16(1+\beta^2)^2k^4+(1+\alpha\beta)^2\omega^2]^2}-\frac{[2\beta^2k^2+\omega^2(1+\alpha\beta)\eta]}{2(1+\beta^2)k^2\omega^2}\right\},$$

$$\rho_2=\frac{(1+\alpha\beta)(-1-\alpha^2+4\alpha\beta-5\beta^2+3\alpha^2\beta^2-4\alpha\beta^3)}{4(1+\beta^2)^2k^2}.$$

We see that, as expected, the lowest nonvanishing order of the real part of $s^{\pm}$ is $O(\delta^4)$ and is independent of the $\pm$. The quantity σ_4 thus determines the solution stability, and in particular the shift $\eta_{\rm EBF}$ in the generalized Eckhaus boundary.

These results may be compared with those for $m=\delta\tilde{m}$ and an unmodulated control parameter $\mu=[(3+\alpha\beta+2\beta^2)/(1+\alpha\beta)]k^2+\delta^2\eta$ in the limit $\delta\to 0$. In this case, the growth rates for $a^{\pm}$ are

$$s^+=s^-=\delta\sigma_1^u+\delta^3\sigma_3^u+\delta^4\sigma_4^u+O(\delta^5),$$

where

$$\sigma_1^u=\sigma_1, \quad \sigma_3^u=-i\frac{(1+\alpha\beta)(\beta-\alpha)\tilde{m}^3}{(1+\beta^2)k},$$

$$\sigma_4^u=\rho_1^u\tilde{m}^2+\rho_2^u\tilde{m}^4,$$

and

$$\rho_1^u=-\frac{(1+\alpha\beta)^2\eta}{2(1+\beta^2)k^2}, \quad \rho_2^u=\rho_2.$$

In particular, these growth rates are equal to the growth rates for the modulated case in the limit that the modulation frequency $\omega\to\infty$. Since σ_4^u determines the stability and since ρ_1^u and ρ_2^u may be positive or negative depending on the values of η, α, and β, the dependence of σ_4^u on $\tilde{m}$ is completely analogous to the modulated case. In both cases the singular behavior of the growth rate as $k\to 0$ is a consequence of our assumption that $m\ll k$, which forms the basis of our expansion scheme.

There are four generic cases for the dependence of σ_4 on $\tilde{m}$ corresponding to the four possible combinations of signs of ρ_1 and ρ_2. Specifically, (a) if $\rho_2<0$ and $\rho_1<0$, then $\sigma_4<0$ for all $\tilde{m}$. (b) If $\rho_2<0$ and $\rho_1>0$, then $\sigma_4>0$ for $0<\tilde{m}<\tilde{m}_u$, and $\sigma_4<0$ for $\tilde{m}>\tilde{m}_u$, where

$$\tilde{m}_u=\left(-\frac{\rho_1}{\rho_2}\right)^{1/2}. \tag{18}$$

Also, the value of $\tilde{m}$ corresponding to the maximum growth rate is

$$\tilde{m}_g\equiv\left(-\frac{\rho_1}{2\rho_2}\right)^{1/2}=\frac{1}{\sqrt{2}}\tilde{m}_u.$$

(c) If $\rho_2>0$ and $\rho_1<0$, then $\sigma_4<0$ for $0<\tilde{m}<\tilde{m}_u$, and $\sigma_4>0$ for $\tilde{m}>\tilde{m}_u$, where $\tilde{m}_u$ is again given by Eq. (18). (d) If $\rho_2>0$ and $\rho_1>0$, then $\sigma_4>0$ for all $\tilde{m}$.

Note that the sign of ρ_2 is independent of k. Thus for α and β such that $\rho_2<0$ (see Fig. 1) the modulational instability first occurs in the limit of small $\tilde{m}$, and its threshold is then given by $\rho_1=0$, or equivalently, $\eta=\eta_{\rm EBF}$, where

$$\eta_{\rm EBF}=\frac{64(1+\alpha\beta)(1+\beta^2)^2k^6}{[16(1+\beta^2)^2k^4+(1+\alpha\beta)^2\omega^2]^2}-\frac{2\beta^2k^2}{(1+\alpha\beta)\omega^2}. \tag{19}$$

Typical growth rates as a function of $\tilde{m}$ are shown in Fig. 2.

In the original variables the generalized Eckhaus boundary takes the explicit form

$$\mu=\mu_{\rm EBF}(k;\omega)\equiv\tilde{\mu}_0+\delta^2\eta_{\rm EBF} \tag{20}$$

and may lie inside or outside the original boundary $\mu=\tilde{\mu}_0$ depending on whether $\eta_{\rm EBF}>0$ or $\eta_{\rm EBF}<0$. Thus the band of stable wave numbers may decrease or increase (or remain unchanged) depending on the values of α, β, and ω. It is easiest to determine which of these applies by plotting $\tilde{\mu}_0$ and $\mu_{\rm EBF}$ versus k for the parameters of interest (see Fig. 3). An important special case for which $\rho_2<0$ is $\alpha=\beta=0$, which gives

$$\mu_E(k;\omega)=3k^2+\frac{64k^6\delta^2}{(16k^4+\omega^2)^2} \tag{21}$$

[see Fig. 4(a)]. In this case the band of stable wave numbers is always reduced, with lower frequencies giving greater reduction. For $\rho_2>0$ [see Fig. 4(b)], all solutions of the form (2) are unstable, although the instability sets in for finite $\tilde{m}$ if $\rho_1<0$. In this case the expansion is not able to capture the most unstable wave numbers although we expect stabilization for large enough m; in particular, there is no *resonant* sideband instability with $O(1)$ wave numbers excited in response to $O(\delta)$ modulation of the control parameter because the $O(\delta)$ growth rate $\hat{s}$ of $c^{\pm}$ due to such a resonance cannot compete with the $O(1)$ decay rate $\tilde{s}$. Note, finally, that in the limit $\omega\to 0$ the Eckhaus boundary $\eta=\eta_{\rm EBF}$ does not reduce to the condition $\eta_{\rm EBF}=0$ appropriate to the unmodulated problem ($\tilde{\mu}_1=0$); the multiple scales expansion breaks

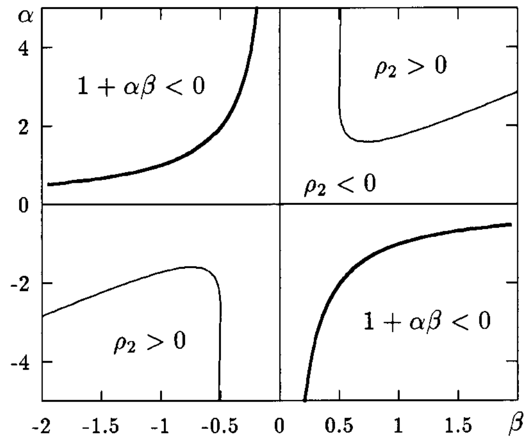

FIG. 1. The sign of ρ_2 in the (α,β) plane. Only the region $1+\alpha\beta>0$ is of interest; if $1+\alpha\beta<0$ no Eckhaus boundary is present.

down in this limit. However, as in other problems of this type, e.g., [33], one does *not* expect to recover in this limit the results for the unmodulated problem. In particular, when the modulation frequency is low, the Eckhaus boundary is shifted quasistatically. Since the growth rate of the instability for finite δ is finite, the instability will be triggered and have time to grow to finite amplitude before the sign of $\mu-\tilde{\mu}_0$ reverses. Thus in the limit $\omega\rightarrow 0$ we expect the generalized Eckhaus boundary to be shifted by $O(\delta)$, and not $O(\delta^2)$, i.e., by a substantially larger amount than in the finite ω case. The singular behavior of $\eta_{\rm EBF}$ found above as $\omega\rightarrow 0$ supports this expectation, but suggests that the physical argument requires modification when $\beta=0$.

IV. CONCLUSION

In this paper, we have described a general method for obtaining the effect of modulation of the control parameter on the generalized Eckhaus boundary, with emphasis on the case when the control parameter is undergoing asymptotically small sinusoidal oscillations about a constant value. For this example, it was found that the band of stable wave vectors may be expanded or reduced depending on the values of α and β, while in the special case $\alpha=\beta=0$ the band is always reduced, with lower frequencies giving greater reduction. We have found no evidence for resonant excitation of the Eckhaus-Benjamin-Feir instability. We anticipate that the evolution of the long wavelength instability (when present) will continue to be governed by the Kuramoto-Sivashinsky equation with σ_4 determining the coefficients of the second and fourth spatial derivatives, while σ_1 and σ_3 contribute drift (first derivatives) and dispersion (third derivatives). Only if the modulation time scale is $O(\delta^{-4})$ will the evolution of the instability be described by the Kuramoto-Sivashinsky equation with a fluctuating control parameter.

The stability results obtained are in qualitative agreement with those of Hernández-García *et al.* [23] on the effects of fluctuations near the Eckhaus boundary in the Swift-Hohenberg equation, who found through direct numerical simulation that the fluctuations decrease the effective width of the Eckhaus-stable band. This conclusion is to be compared with our results for $\alpha=\beta=0$. Although experimental confirmation of this conclusion appears to be lacking, the $\alpha=\beta=0$ case could be tested in Rayleigh-Bénard convection with either oscillatory temperature of the lower and/or upper plates or oscillating gravitational acceleration. A similar experiment on Taylor vortex flow with the modulation arising from oscillations in the angular speed of the inner cylinder appears feasible. The effect on the generalized Eckhaus boundary of combined periodic modulation and either additive stochastic noise or noise in the control parameter itself is also of interest, cf. [32].

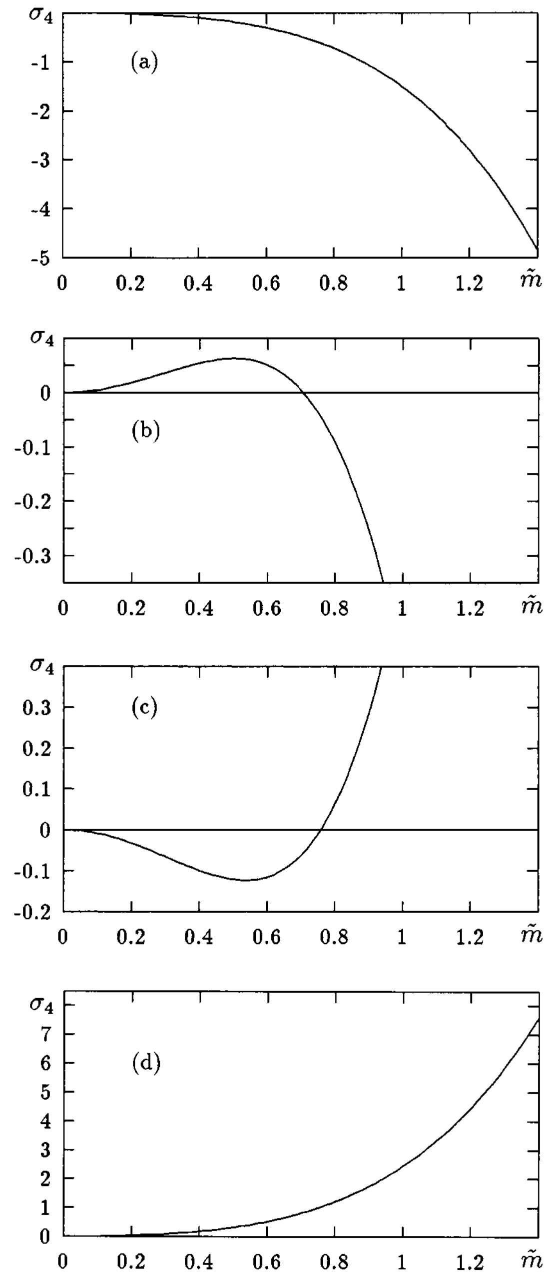

FIG. 2. The growth rate σ_4 as a function of the perturbation wave number $\tilde{m}$, showing the four generic possibilities. Here $\omega=1$, $k=0.5$ and (a) $\alpha=\beta=0$, $\eta=0.5$ $(\rho_1<0,\ \rho_2<0)$, (b) $\alpha=\beta=0$, $\eta=0$ $(\rho_1>0,\ \rho_2<0)$, (c) $\alpha=2$, $\beta=1$, $\eta=0$ $(\rho_1<0,\ \rho_2>0)$, and (d) $\alpha=2$, $\beta=1$, $\eta=-0.2$ $(\rho_1>0,\ \rho_2>0)$.

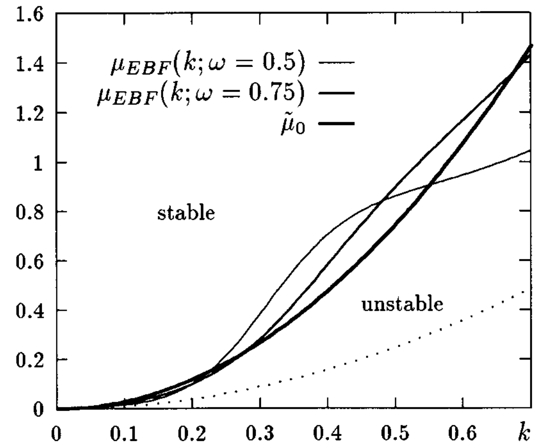

FIG. 3. An example of the frequency dependence of the generalized Eckhaus boundary with sinusoidal modulation where $\alpha = 0.5$ and $\beta = 0.5$ (so that $\rho_2 < 0$). Notice, for example, that for $\tilde{\mu}_0 = 1$, the band of stable wave numbers is increased for $\omega = 0.5$ and reduced for $\omega = 0.75$ with respect to the unmodulated Eckhaus boundary. The dotted line is the boundary for the primary instability. This figure is plotted with $\delta = 1$ for clarity.

ACKNOWLEDGMENTS

This work was supported in part by the National Science Foundation under Grant No. DMS-9406144.

APPENDIX A: EXPANSION OF $p(t)$

The terms in the expansion (12) are found by expanding $q(t) \equiv 1/p(t)$ as

$$q(t) = \tilde{q}_0 + \delta\tilde{q}_1(t) + \delta^2\tilde{q}_2(t) + \cdots,$$

solving Eq. (6) order by order in the long-time limit, and expressing the results in terms of the $\tilde{p}_i(t)$'s. This gives

$$\tilde{p}_0 = \tilde{\mu}_0 - k^2$$

and

$$\tilde{p}_i(t) = r_i(t) + \frac{2\tilde{p}_0^2}{g(t)}\int^t h_i(t')g(t')dt',$$

where

$$g(t) = \exp[2(\tilde{\mu}_0 - k^2)t]$$

and $r_i(t)$ and $h_i(t)$ depend on $\tilde{p}_{i-1}(t), \tilde{p}_{i-2}(t), \ldots, \tilde{p}_0$. For reference, we give the following formulas for $r_i(t)$ and $h_i(t)$, $i = 1,2,3,4$:

$$r_1(t) = 0,$$

$$r_2(t) = \frac{[\tilde{p}_1(t)]^2}{\tilde{p}_0},$$

$$r_3(t) = \frac{2\tilde{p}_1(t)\tilde{p}_2(t)}{\tilde{p}_0} - \frac{2[\tilde{p}_1(t)]^3}{\tilde{p}_0^2},$$

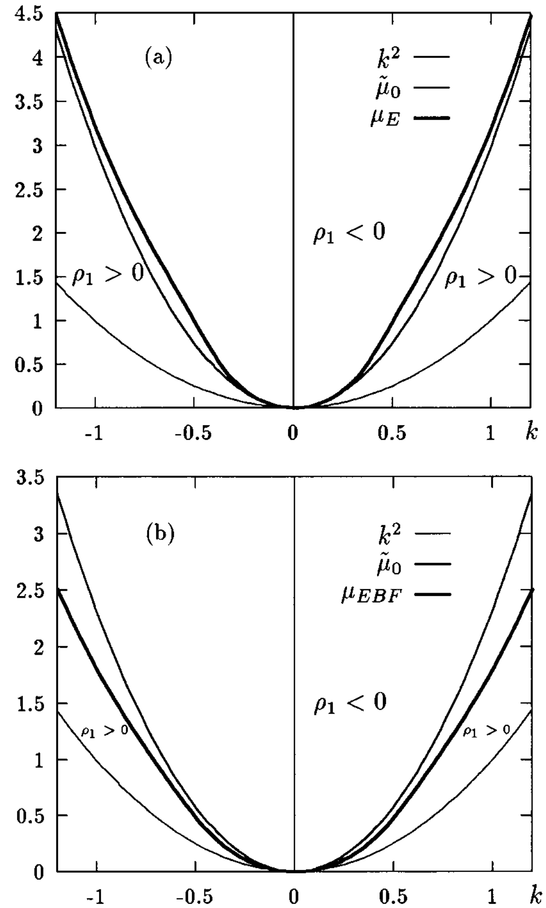

FIG. 4. (a) The Eckhaus boundary in the system with external sinusoidal modulation (heavy line) compared with the boundary in the absence of modulation (intermediate thickness line) and the region of primary instability (thin line). The parameters are $\omega = 1$, and $\alpha = \beta = 0$ ($\rho_2 < 0$), corresponding to the real Ginzburg-Landau equation with modulation. (b) $\omega = 1$, $\alpha = 2$, $\beta = 1$ ($\rho_2 > 0$). Here the heavy line does not represent the Eckhaus boundary because all solutions of the form (2) are unstable, although for $\rho_1 < 0$ the instability sets in at finite $\tilde{m}$ (see text). The signs of ρ_1 are indicated and $\delta = 1$ for clarity.

$$r_4(t) = \frac{[\tilde{p}_1(t)]^4}{\tilde{p}_0^3} - \frac{3[\tilde{p}_1(t)]^2\tilde{p}_2(t)}{\tilde{p}_0^2} + \frac{[\tilde{p}_2(t)]^2 + 2\tilde{p}_1(t)\tilde{p}_3(t)}{\tilde{p}_0},$$

$$h_1(t) = \frac{\tilde{\mu}_1(t)}{\tilde{p}_0},$$

$$h_2(t) = \frac{\tilde{\mu}_2(t)}{\tilde{p}_0} - \frac{\tilde{\mu}_1(t)\tilde{p}_1(t)}{\tilde{p}_0^2},$$

$$h_3(t) = \frac{\tilde{\mu}_3(t)}{\tilde{p}_0} - \frac{\tilde{\mu}_2(t)\tilde{p}_1(t)}{\tilde{p}_0^2} + \tilde{\mu}_1(t)\left(\frac{[\tilde{p}_1(t)]^2}{\tilde{p}_0^3} - \frac{\tilde{p}_2(t)}{\tilde{p}_0^2}\right),$$

$$h_4(t)=\frac{\tilde{\mu}_4(t)}{\tilde{p}_0}-\frac{\tilde{\mu}_3(t)\tilde{p}_1(t)}{\tilde{p}_0^2}+\tilde{\mu}_2(t)\left(\frac{[\tilde{p}_1(t)]^2}{\tilde{p}_0^3}-\frac{\tilde{p}_2(t)}{\tilde{p}_0^2}\right)+\tilde{\mu}_1(t)\left(-\frac{[\tilde{p}_1(t)]^3}{\tilde{p}_0^4}+\frac{2\tilde{p}_1(t)\tilde{p}_2(t)}{\tilde{p}_0^3}-\frac{\tilde{p}_3(t)}{\tilde{p}_0^2}\right).$$

APPENDIX B: MULTIPLE SCALES ANALYSIS OF EQ. (13)

In terms of the notation $D_i \equiv \partial/\partial T_i$ the multiple scale analysis of Eq. (13) leads to the following hierarchy of equations:

$$D_0^2 c_0^{\pm}+f_0^{\pm}c_0^{\pm}=0,$$

$$D_0^2 c_1^{\pm}+f_0^{\pm}c_1^{\pm}=-f_1^{\pm}c_0^{\pm}-2D_1D_0c_0^{\pm},$$

$$D_0^2 c_2^{\pm}+f_0^{\pm}c_2^{\pm}=-f_1^{\pm}c_1^{\pm}-f_2^{\pm}c_0^{\pm}-D_1^2c_0^{\pm}-2D_2D_0c_0^{\pm}-2D_1D_0c_1^{\pm},$$

$$D_0^2 c+f_0^{\pm}c_3^{\pm}=-f_1^{\pm}c_2^{\pm}-f_2^{\pm}c_1^{\pm}-f_3^{\pm}c_0^{\pm}-D_1^2c_1^{\pm}-2D_3D_0c_0^{\pm}-2D_2D_1c_0^{\pm}-2D_2D_0c_1^{\pm}-2D_1D_0c_2^{\pm},$$

$$D_0^2 c_4^{\pm}+f_0^{\pm}c_4^{\pm}=-f_1^{\pm}c_3^{\pm}-f_2^{\pm}c_2^{\pm}-f_3^{\pm}c_1^{\pm}-f_4^{\pm}c_0^{\pm}-D_1^2c_2^{\pm}-D_2^2c_0^{\pm}-2D_4D_0c_0^{\pm}-2D_3D_1c_0^{\pm}-2D_3D_0c_1^{\pm}-2D_2D_1c_1^{\pm}-2D_2D_0c_2^{\pm}-2D_0D_1c_3^{\pm}.$$

These are obtained by equating terms at $O(1), O(\delta), \ldots, O(\delta^4)$, respectively.

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