Multi-group disease models

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man to the bonds of Hell.

with the devil to darken the spirit and continue empty prophecies. The danger already exists among mathematicians and all who make

—St. Augustine
Revisited

Number, affected is $\Lambda = S - I = I^{0}/R^{0}$.

Thus

$S^{0}/I = 1$

Probability of transmission $\phi$
Contact Rate $c$
Average Duration of infection $D$
Probability of transmission $\phi$

$R^{0} = cD$

In a susceptible population,

individual in a susceptible population

$R^{0}$ is the number of people who would be infected by an infectious

$R^{0}$ Revisited
Proportion malarious

Effective Biting Rate

Ross (1908)
The Endemic Curve

Proportion affected

$R_0$
Host Heterogeneity

• Demographic Heterogeneity
  • Spatial Heterogeneity
    - Contact Rate
    - Transmission
    - Susceptibility
  • Parametric Heterogeneity
USA Tuberculosis incidence by zip code
A multi-group gonorrhea model

Assume people are differentiated only by mixing rate, and rescale so that the effective mixing rate is equivalent to the subgroup

\[ (q) N q(p, q) d = (p) N p(q, p) d \]

and

\[ I = q p(q, p) d \int \]

The mixing function \( p(a; b) \) must satisfy:

\[ Z p(a; b) dB = 1 \]

where

\[ q p (\frac{q}{q}) N (q, p) d \int = (p) V \]

is the proportion of group \( a \)'s contacts that are infectious.

\[ (p) I - (p) V ((p) I - (p) N) p = (p) I \]

reproductive number.
A multi-group gonorrhea model

The parameters here are the distribution $N(a)$ and the mixing functions $d(q,a)$.

It will often be more useful to try to estimate the moments of these $R_0$ transitions rather than the functions themselves, and to relate $R_0$ and the proportion affected to these moments (using approximate relationships).

The parameters here are the distribution $N(a)$ and the mixing functions $d(q,a)$.
If people mix randomly, then \( p(a;b) \) will depend only on subgroup \( b \)'s importance in the mixing pool:

\[
p(a;b) = \frac{bN}{ZcN} dc
\]

The model becomes:

\[
(\nu)I - V((\nu)I - (\nu)N)\nu = (\nu)I
\]

where \( V \) is now the constant mixing-weighted proportion of the population that is infectious:

\[
V = Z_a I(\nu) \int d\nu
\]

\[
(\nu)N(\nu) = (q, \nu) d\nu
\]

If people mix randomly, then \( \nu \)'s importance in the mixing pool will depend only on subgroup importance.

Random mixing
Next-generation framework

By inspection,

\[(\nu) N \nu = (\nu) I\]

In the random-mixing case, \(V\) is a constant, so

\[(\nu) V ((\nu) I - (\nu) N) \nu = (\nu) I\]

We can also use a next-generation approach for the equilibrium:

\[(\nu) V (\nu) N \nu = (\nu) I^{0} R\]

In this generation, the operator that takes the distribution of cases in the next generation to the distribution of cases in the next generation is the eigenvalue of the operator that takes the distribution of cases in this generation to the distribution of cases in the next generation.
mean and $\sigma$ of the distribution $\mathcal{N}(\mu, \sigma)$, where $\mu$ and $\sigma$ are the mean and $\sigma$ of the subgroup reproductive numbers. We can also write $R_0 = \left(1 + \frac{\sigma^2}{\mu^2}\right) R_0$, where $\mu$ and $\sigma$ are the mean and $\sigma$ of the distribution $\mathcal{N}(\mu, \sigma)$.

This is the mixing-weighted average of the subgroup reproductive numbers.

\[
\frac{\int np(v) \mathcal{N}(v) dv}{\int np(v) \mathcal{I}(v) dv} = R_0
\]

\[
\frac{\int np(v) \mathcal{N}(v) dv}{\int np(v) \mathcal{I}(v) dv} = V
\]

Recalling: Next-generation framework
The Effect of Variation in Sexual Mixing Rate

Prop.infected

CV = 0.0

R_0

0 0.2 0.4 0.6 0.8 1

8 4 2 1

0 0.2 0.4 0.6 0.8 1

The Effect of Variation in Sexual Mixing Rate
The Effect of Variation in Sexual Mixing Rate

The figure illustrates the impact of variation in sexual mixing rate on the proportion of individuals affected. The x-axis represents the mean mixing rate, while the y-axis shows the proportion affected. Different lines indicate the CV (Coefficient of Variation) values of 0.0, 0.5, 1.0, and 2.0. As the mean mixing rate increases, the proportion affected also increases, with the effect being more pronounced at higher CV values.
Assortative mixing

\[ V(d - 1) + \frac{(p)N}{(p)I}d = (p)V \]

\[ \text{Thus} \]

\[ \frac{\varepsilon p(q)N\varepsilon}{(q)Nq} \int (d - 1) + (q \cdot q)d = (q \cdot q)d \]

bad approximation: bizarre construct of "preferred mixing" which is not necessarily a

Assortative mixing is usually approximated with the somewhat
Assortative mixing satisfies
\[ \mathcal{L}^0 \]
and we can calculate a threshold quantity
\[ (d - 1) \int (d - 1) \]

\[ \frac{\nu p(v) N v}{\nu p (\frac{d v - 1}{h(v) N z v})} \int (d - 1) \]

\[ \frac{\nu p(v) N v}{\nu p (\frac{d v - 1}{h(v) N z v})} \int (d - 1) \]

\[ \text{Assortative mixing} \]
What attests the endemic curve? (random mixing)

Homogeneous:

\[ R_0 = Dc \]

\[ S = 1 \]

\[ R_0 = 0 \]

Heterogeneous:

\[ S > 0 \]

\[ S < 0 \]

\[ S = a \]

\[ R_0 = \frac{0}{1} \]

\( ^a \theta = 0 \)

\( ^0 \theta = 1 \)

\( ^a \theta = 0 \)
What else flattens the endemic curve?

Infected people's sexual partners are more likely than average to be infected, leading to wasted contacts (from the point of view of the disease). The tendency of people to mix with people who are in similar neighborhoods or social groups.

**Deterministic Assortative Mixing** The tendency of people to mix with people who are in the same neighborhoods or social groups.

**Stochastic Demographic Effects** The tendency of people to infect people who are in similar neighborhoods or social groups.