Appendix

Partial-Fraction Expansion

Before we present MATLAB approach to the partial-fraction expansions of transfer functions, we discuss the manual approach to the partial-fraction expansions of transfer functions.

Partial-Fraction Expansion when \( F(s) \) Involves Distinct Poles Only. Consider \( F(s) \) written in the factored form

\[
F(s) = \frac{B(s)}{A(s)} = \frac{K(s + z_1)(s + z_2) \cdots (s + z_m)}{(s + p_1)(s + p_2) \cdots (s + p_n)}, \quad \text{for } m < n
\]

where \( p_1, p_2, \ldots, p_n \) and \( z_1, z_2, \ldots, z_m \) are either real or complex quantities, but for each complex \( p_j \) or \( z_j \) there will occur the complex conjugate of \( p_j \) or \( z_j \), respectively. If \( F(s) \) involves distinct poles only, then it can be expanded into a sum of simple partial fractions as follows:

\[
F(s) = \frac{B(s)}{A(s)} = \frac{a_1}{s + p_1} + \frac{a_2}{s + p_2} + \cdots + \frac{a_n}{s + p_n} \quad \text{(B-1)}
\]

where \( a_k \) \((k = 1, 2, \ldots, n)\) are constants. The coefficient \( a_k \) is called the residue at the pole at \( s = -p_k \). The value of \( a_k \) can be found by multiplying both sides of Equation (B-1) by \((s + p_k)\) and letting \( s = -p_k \), which gives

\[
\left[ B(s) \right]_{s=-p_k} A(s) = \left[ \frac{a_1}{s + p_1} (s + p_k) + \frac{a_2}{s + p_2} (s + p_k) + \cdots + \frac{a_n}{s + p_n} (s + p_k) \right]_{s=-p_k}
\]

\[
= a_k
\]
We see that all the expanded terms drop out with the exception of \(a_k\). Thus the residue \(a_k\) is found from
\[
a_k = \left[ \frac{(s + p_k) B(s)}{A(s)} \right]_{s = -p_k}
\]
Note that, since \(f(t)\) is a real function of time, if \(p_1\) and \(p_2\) are complex conjugates, then the residues \(a_1\) and \(a_2\) are also complex conjugates. Only one of the conjugates, \(a_1\) or \(a_2\), needs to be evaluated, because the other is known automatically.

Since
\[
\mathcal{L}^{-1} \left[ \frac{a_k}{s + p_k} \right] = a_k e^{-p_k t}
\]
\(f(t)\) is obtained as
\[
f(t) = \mathcal{L}^{-1}[F(s)] = a_1 e^{-p_1 t} + a_2 e^{-p_2 t} + \cdots + a_n e^{-p_n t}, \quad \text{for } t \geq 0
\]

**EXAMPLE B-1** Find the inverse Laplace transform of
\[
F(s) = \frac{s + 3}{(s + 1)(s + 2)}
\]
The partial-fraction expansion of \(F(s)\) is
\[
F(s) = \frac{s + 3}{(s + 1)(s + 2)} = \frac{a_1}{s + 1} + \frac{a_2}{s + 2}
\]
where \(a_1\) and \(a_2\) are found as
\[
a_1 = \left[ \frac{(s + 1) s + 3}{(s + 1)(s + 2)} \right]_{s = -1} = \left[ \frac{s + 3}{s + 2} \right]_{s = -1} = 2
\]
\[
a_2 = \left[ \frac{(s + 2) s + 3}{(s + 1)(s + 2)} \right]_{s = -2} = \left[ \frac{s + 3}{s + 1} \right]_{s = -2} = -1
\]
Thus
\[
f(t) = \mathcal{L}^{-1}[F(s)]
\]
\[
= \mathcal{L}^{-1} \left[ \frac{2}{s + 1} \right] + \mathcal{L}^{-1} \left[ \frac{-1}{s + 2} \right]
\]
\[
= 2e^{-t} - e^{-2t}, \quad \text{for } t \geq 0
\]

**EXAMPLE B-2** Obtain the inverse Laplace transform of
\[
G(s) = \frac{s^3 + 5s^2 + 9s + 7}{(s + 1)(s + 2)}
\]
Here, since the degree of the numerator polynomial is higher than that of the denominator polynomial, we must divide the numerator by the denominator.
\[
G(s) = s + 2 + \frac{s + 3}{(s + 1)(s + 2)}
\]
Note that the Laplace transform of the unit-impulse function \( \delta(t) \) is 1 and that the Laplace transform of \( d\delta(t)/dt \) is \( s \). The third term on the right-hand side of this last equation is \( F(s) \) in Example B–1. So the inverse Laplace transform of \( G(s) \) is given as

\[
g(t) = \frac{d}{dt} \delta(t) + 2d(t) + 2e^t - e^{-2t}, \quad \text{for } t \geq 0
\]

**EXAMPLE B–3** Find the inverse Laplace transform of

\[
F(s) = \frac{2s + 12}{s^2 + 2s + 5}
\]

Notice that the denominator polynomial can be factored as

\[
s^2 + 2s + 5 = (s + 1 + j2)(s + 1 - j2)
\]

If the function \( F(s) \) involves a pair of complex-conjugate poles, it is convenient not to expand \( F(s) \) into the usual partial fractions but to expand it into the sum of a damped sine and a damped cosine function.

Noting that \( s^2 + 2s + 5 = (s + 1)^2 + 2^2 \) and referring to the Laplace transforms of \( e^{-\alpha t} \sin \omega t \) and \( e^{-\alpha t} \cos \omega t \), rewritten thus,

\[
\mathcal{L}[e^{-\alpha t} \sin \omega t] = \frac{\omega}{(s + \alpha)^2 + \omega^2}
\]

\[
\mathcal{L}[e^{-\alpha t} \cos \omega t] = \frac{s + \alpha}{(s + \alpha)^2 + \omega^2}
\]

the given \( F(s) \) can be written as a sum of a damped sine and a damped cosine function:

\[
F(s) = \frac{2s + 12}{s^2 + 2s + 5} = \frac{10 + 2(s + 1)}{(s + 1)^2 + 2^2}
\]

\[
= \frac{5}{(s + 1)^2 + 2^2} + 2 \cdot \frac{s + 1}{(s + 1)^2 + 2^2}
\]

It follows that

\[
f(t) = \mathcal{L}^{-1}[F(s)]
\]

\[
= 5e^{-t} \sin 2t + 2e^{-t} \cos 2t, \quad \text{for } t \geq 0
\]

**Partial-Fraction Expansion when \( F(s) \) Involves Multiple Poles.** Instead of discussing the general case, we shall use an example to show how to obtain the partial-fraction expansion of \( F(s) \).

Consider the following \( F(s) \):

\[
F(s) = \frac{s^2 + 2s + 3}{(s + 1)^3}
\]

The partial-fraction expansion of this \( F(s) \) involves three terms,

\[
F(s) = \frac{B(s)}{A(s)} = \frac{b_1}{s + 1} + \frac{b_2}{(s + 1)^2} + \frac{b_3}{(s + 1)^3}
\]
where $b_3$, $b_2$, and $b_1$ are determined as follows. By multiplying both sides of this last equation by $(s + 1)^3$, we have

\[(s + 1)^3 \frac{B(s)}{A(s)} = b_1(s + 1)^2 + b_2(s + 1) + b_3 \quad (B-2)\]

Then letting $s = -1$, Equation (B-2) gives

\[\left[(s + 1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_3\]

Also, differentiation of both sides of Equation (B-2) with respect to $s$ yields

\[\frac{d}{ds}\left[(s + 1)^3 \frac{B(s)}{A(s)}\right] = b_2 + 2b_1(s + 1) \quad (B-3)\]

If we let $s = -1$ in Equation (B-3), then

\[\frac{d}{ds}\left[(s + 1)^3 \frac{B(s)}{A(s)}\right]_{s=-1} = b_2\]

By differentiating both sides of Equation (B-3) with respect to $s$, the result is

\[\frac{d^2}{ds^2}\left[(s + 1)^3 \frac{B(s)}{A(s)}\right] = 2b_1\]

From the preceding analysis it can be seen that the values of $b_3$, $b_2$, and $b_1$ are found systematically as follows:

\[b_3 = \left[(s + 1)^3 \frac{B(s)}{A(s)}\right]_{s=-1}\]
\[= (s^2 + 2s + 3)_{s=-1}\]
\[= 2\]

\[b_2 = \left\{\frac{d}{ds}\left[(s + 1)^3 \frac{B(s)}{A(s)}\right]\right\} \quad (s=-1)\]
\[= \left[\frac{d}{ds} (s^2 + 2s + 3)\right]_{s=-1}\]
\[= (2s + 2)_{s=-1}\]
\[= 0\]

\[b_1 = \frac{1}{2!}\left\{\frac{d^2}{ds^2}\left[(s + 1)^3 \frac{B(s)}{A(s)}\right]\right\} \quad (s=-1)\]
\[= \frac{1}{2!}\left[\frac{d^2}{ds^2} (s^2 + 2s + 3)\right]_{s=-1}\]
\[= \frac{1}{2} (2) = 1\]
We thus obtain
\[ f(t) = \mathcal{L}^{-1}[F(s)] \]
\[ = \mathcal{L}^{-1}\left[ \frac{1}{s + 1} \right] + \mathcal{L}^{-1}\left[ \frac{0}{(s + 1)^2} \right] + \mathcal{L}^{-1}\left[ \frac{2}{(s + 1)^3} \right] \]
\[ = e^{-t} + 0 + t^2 e^{-t} \]
\[ = (1 + r^2)e^{-t}, \quad \text{for } t \geq 0 \]

**Comments.** For complicated functions with denominators involving higher-order polynomials, partial-fraction expansion may be quite time consuming. In such a case, use of MATLAB is recommended.

**Partial-Fraction Expansion with MATLAB.** MATLAB has a command to obtain the partial-fraction expansion of \( B(s)/A(s) \). Consider the following function \( H(s)/A(s) \):
\[
\frac{B(s)}{A(s)} = \frac{\text{num}}{\text{den}} = \frac{b_0 s^n + b_1 s^{n-1} + \cdots + b_n}{s^a + a_1 s^{a-1} + \cdots + a_n}
\]

where some of \( a_i \) and \( b_i \) may be zero. In MATLAB row vectors \( \text{num} \) and \( \text{den} \) specify the coefficients of the numerator and denominator of the transfer function. That is,
\[
\text{num} = [b_0 \ b_1 \ \cdots \ b_n] \\
\text{den} = [1 \ a_1 \ \cdots \ a_n]
\]

The command
\[
[r,p,k] = \text{residue}(\text{num},\text{den})
\]
finds the residues \( r \), poles \( p \), and direct terms \( k \) of a partial-fraction expansion of the ratio of two polynomials \( B(s) \) and \( A(s) \).

The partial-fraction expansion of \( B(s)/A(s) \) is given by
\[
\frac{B(s)}{A(s)} = \frac{r(1)}{s - p(1)} + \frac{r(2)}{s - p(2)} + \cdots + \frac{r(n)}{s - p(n)} + k(s) \quad \text{(B-4)}
\]

Comparing Equations (B-1) and (B-4), we note that \( p(1) = -p_1, p(2) = -p_2, \ldots, p(n) = -p_n; r(1) = a_1, r(2) = a_2, \ldots, r(n) = a_n \); \( [k(s) \text{ is a direct term.}] \)

**EXAMPLE B-4** Consider the following transfer function,
\[
\frac{B(s)}{A(s)} = \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6}
\]
For this function,
\[
\begin{align*}
\text{num} &= [2 \ 5 \ 3 \ 6] \\
\text{den} &= [1 \ 6 \ 11 \ 6]
\end{align*}
\]
The command
\[
[r,p,k] = \text{residue(num,den)}
\]
gives the following result:
\[
\begin{align*}
[r,p,k] &= \text{residue(num,den)} \\
r &= \\
&= -6.0000 \\
&= -4.0000 \\
&= 3.0000 \\
p &= \\
&= -3.0000 \\
&= -2.0000 \\
&= -1.0000 \\
k &= 2
\end{align*}
\]
(Note that the residues are returned in column vector \( r \), the pole locations in column vector \( p \), and the direct term in row vector \( k \).) This is the MATLAB representation of the following partial-fraction expansion of \( B(s)/A(s) \):
\[
\frac{B(s)}{A(s)} = \frac{2s^3 + 5s^2 + 3s + 6}{s^3 + 6s^2 + 11s + 6} = \frac{-6}{s + 3} + \frac{-4}{s + 2} + \frac{3}{s + 1} + 2
\]
Note that if \( p(j) = p(j + 1) = \cdots = p(j + m - 1) \) (that is, \( p_j = p_{j+1} = \cdots = p_{j+m-1} \)), the pole \( p(j) \) is a pole of multiplicity \( m \). In such a case, the expansion includes terms of the form
\[
\frac{r(j)}{s - p(j)} + \frac{r(j + 1)}{(s - p(j))^2} + \cdots + \frac{r(j + m - 1)}{(s - p(j))^m}
\]
For details, see Example B-5.
**EXAMPLE B-5** Expand the following \( \frac{B(s)}{A(s)} \) into partial fractions with MATLAB.

\[
\frac{B(s)}{A(s)} = \frac{s^2 + 2s + 3}{(s + 1)^3} = \frac{s^2 + 2s + 3}{s^3 + 3s^2 + 3s + 1}
\]

For this function, we have

\[
\text{num} = [1 \ 2 \ 3] \\
\text{den} = [1 \ 3 \ 3 \ 1]
\]

The command

\[
[r,p,k] = \text{residue(num,den)}
\]

gives the result shown next:

\[
\begin{align*}
\text{num} &= [1 \ 2 \ 3] ; \\
\text{den} &= [1 \ 3 \ 3 \ 1] ; \\
[r,p,k] &= \text{residue(num,den)} \\
\[r = \]
1.0000 \\
0.0000 \\
2.0000 \\
\[p = \]
-1.0000 \\
-1.0000 \\
-1.0000 \\
k &= []
\end{align*}
\]

It is the MATLAB representation of the following partial-fraction expansion of \( \frac{B(s)}{A(s)} \):

\[
\frac{B(s)}{A(s)} = \frac{1}{s + 1} + \frac{0}{(s + 1)^2} + \frac{2}{(s + 1)^3}
\]

Note that the direct term \( k \) is zero.