Introduction to Koopman operator theory of dynamical systems

Hassan Arbabi
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These notes provide a brief introduction to the theory of the Koopman operator. This theory is an alternative operator-theoretic formalism of dynamical systems theory which offers great utility in analysis and control of nonlinear and high-dimensional systems using data. The style of presentation is informal, and a detailed list of references is given at the end which contain rigorous mathematical treatments as well as a variety of applications to physics and engineering problems.

0.1 Classical theory of dynamical systems

A dynamical system, in the abstract sense, consists of two things: a set of states through which we can describe the evolution of a system, and a rule for that evolution. Although this viewpoint is very general and may be applied to almost any system that evolves with time, often the fruitful and conclusive results are achievable when we pose some mathematical structure on the dynamical system, for example, we often assume the
set of states form a linear space with nice geometric properties and the rule of evolution
has some order of regularity on that space. The prominent examples of such dynamical
systems are amply found in physics, where we use differential equations to describe the
physical variables change in time. In this note, we specially focus on dynamical systems
that can be represented as

\[ \dot{x} = f(x), \]  

(0.1)

where \( x \) is the state, an element of the state space \( S \subset \mathbb{R}^n \), and \( f : S \to \mathbb{R}^n \) is a vector
field on the state space. Occasionally, we will specify some regularity conditions for \( f \) like
being smooth or a few times differentiable. We also consider dynamical systems given by
the discrete-time map

\[ x^{t+1} = T(x^t), \quad t \in \mathbb{Z} \]  

(0.2)

where \( x \) belongs to the state space \( S \subset \mathbb{R}^n \), \( T : S \to S \) is the dynamic map and \( t \) is the
discrete time index. Just like the continuous-time system in (0.1), we may need to make
some extra assumptions on \( T \). The discrete time representation of dynamical system does
not often show up in physical systems, but we can use it to represent continuous-time
systems, for example, through discrete-time sampling. This representation also has the
benefit of being more practical because the data collected from dynamical systems almost
always comes in discrete-time samples.

The study of the dynamical systems in (0.1) and (0.2) was dominated by the geometric
viewpoint in much of last century. In this viewpoint, originally due to Henri Poincaré, the
qualitative properties of the solution curves in the state space are studied using geometric
tools and the emphasis is put on the subsets of the state space that play a big role in
the asymptotic behavior of the trajectories. We briefly describe some concepts from this theory here, but a more comprehensive exposition can be found in [1, 2].

Assuming that the solution to (0.1) exists, we define the flow map \( F^t : S \rightarrow S \) to be the map that takes the initial state to the state at time \( t \in \mathbb{R} \), i.e.,

\[
F^t(x_0) = x_0 + \int_{x_0,t'=0}^t f(x(t')) dt'.
\]  

(0.3)

The flow map satisfies the semi-group property, i.e., for every \( s, t \geq 0 \),

\[
F^t \circ F^s(x_0) = F^{s}(x_0) + \int_{x_0,t'=0}^{t} f(x(t')) dt',
\]

\[
= \int_{x_0,t'=0}^{F^{s}(x_0), t'=0} f(x(t')) dt' + \int_{F^{s}(x_0), t'=0}^{t} f(x(t')) dt',
\]

\[
= \int_{x_0,t'=0}^{t+s} f(x(t')) dt',
\]

\[
= F^{t+s}(x_0).
\]  

(0.4)

where \( \circ \) is the composition operator.

Some of the important geometric objects in the state space of continuous-time dynamical systems are as follows:

**Fixed point:** Any point \( x \) in the state space such that \( f(x) = 0 \) (or \( F^t(x) = x \)). The fixed points correspond to the equilibria of physical systems. An important notion with regard to fixed points is the stability, that is whether the trajectories starting in some neighborhood of fixed point stay in its neighborhood over time or not.

**Limit cycle:** Limit cycles are (isolated) closed curves in the state space which correspond to the time-periodic solutions of (0.1). The generalized version of limit cycles are tori (cartesian products of circles) which are associated with quasi-periodic motion.

**Invariant set:** An invariant set \( B \) in the state space satisfies \( F^t(B) \subseteq B \) for all \( t \), i.e.,
the trajectories starting in $B$ remain in $B$. The invariant sets are important because we can isolate the study the dynamics on them from the rest of state space, and they include important objects such as fixed points, limit cycles, attractors and invariant manifolds.

**Attractor:** An attractor is an attracting set with a dense orbit. An attracting set is an invariant subset of the state space to which many initial conditions converge. A dense orbit is a trajectory that comes arbitrarily close to any point on the attracting set. For example, a stable fixed point or limit cycle is an attractor, but union of two separate stable periodic orbits is an attracting set but not an attractor, because the trajectories on one periodic orbit can not come arbitrarily close to the other periodic orbit. A more complicated example of attractor is the famous butterfly-shaped set in the chaotic Lorenz system which is called a strange attractor.

Attractors are the objects that determine the asymptotic (that is post-transient or long-term) dynamics of dissipative dynamical systems. In fact, the mere notion of dissipativity (we can think of it as shrinkage in the state space) is enough to guarantee the existence of an attractor in many systems [3]. In some cases, the state space contains more than one attractor, and the attractors divide the state space into basins of attraction; any point in the basin of attraction of an attractor will converge to it over infinite time.

**Bifurcation:** Bifurcation analysis is the study of changes in the qualitative behavior of all the trajectories due to the changes in vector field $f$ or the map $T$. For example, if we add some forcing term to the vector field $f$, a stable fixed point might turn unstable or a limit cycle might appear out of the blue. A physical example is the evolution of incompressible flows given by Navier-Stokes equations: increasing the Reynolds number may fundamentally change the flow solution from steady to unsteady, or from laminar to turbulent.

Here is the traditional approach to study of dynamical systems: We first discover
or construct a model for the system in the form of (0.1) or (0.2). Sometimes, if we are very lucky, we can come up with analytical (or approximate) solutions and use them to analyze the dynamics, by which, we usually mean finding the attractors, invariant manifolds, imminent bifurcations and so on. A lot of times, this is not possible and we have to use various estimates or approximation techniques to evaluate the qualitative behavior of the system, for example, construct Lyapunov functions to prove the stability of a fixed point. But most of the times, if we want a quantitative analysis or prediction, we have to employ numerical computation and then extract information by looking at a collection of trajectories in the state space.

This approach has contributed the most to our knowledge of dynamical and physical systems around us, but it is falling short in treating the high-dimensional systems that have arisen in various areas of science and technology. A set of classic examples, which regularly arises in physics, is the set of systems that are governed by partial differential equations. In these systems, the state space is infinite-dimensional and the numerical models that we use may have up to billions of degrees of freedom. Some examples of more recent interest include climate system of the earth, smart cars and buildings, power networks, and biological systems with interacting components like neural networks. The first problem with the traditional approach is that simulating the evolution of trajectories for these systems is just devastating due to the large size of the problem. Moreover, unlike the two- or three-dimensional system, the geometric objects in the state space are difficult to realize and utilize. The second problem is the uncertainty in the models or even the sheer lack of a model for simulation or analysis. As a result, the field of dynamical analysis has started shifting toward a less model-based and more data-driven perspective. This shift is also boosted by the increasing amount of data that is produced by today’s powerful computational resources and experimental apparatus. In the next section, we introduce the Koopman operator theory, which is a promising framework for
0.2 The data-driven viewpoint toward dynamical systems and the Koopman operator

In the context of dynamical systems, we interpret the data as knowledge of some variable(s) related to the state of the system. A natural way to put this into the mathematical form is to assume that data is knowledge of variables which are functions of the state. We call these functions observables of the system. Let us discuss an example. The unforced motion of an incompressible fluid inside a box constitutes a dynamical system; one way to realize the state space is to think of it as the set of all smooth velocity fields on the flow domain that satisfy the incompressibility condition. The velocity field changes with time according to a rule of evolution which is the Euler equation. Some examples of observables on this system are pressure/vorticity at a given point in the flow domain, velocity at set of points and the total kinetic energy of the flow. In all these examples, the knowledge of the state, i.e. the velocity field, uniquely determines the value of the observable. We see that this definition allows us to think of the data from most of the flow experiments and simulations as values of observables. We also note that there are some type of data that don’t fit the above definition as an observable of the system. For example, the position of a Lagrangian tracer is not an observable of the above system, since it cannot be determined by mere knowledge of the instantaneous velocity field. In this case, we can alter our dynamical system to include that observable as well: we define a second dynamical system to describe the tracer dynamics (the flow domain is the state space, and the rule of evolution is the time-dependent velocity field). The coupling of these two dynamical systems would explain the all the above observables. So,
determining the notion of observables on a system requires a careful consideration of the underlying processes that affects those observables.

In light of the above discussion, we can formulate the data-driven analysis of dynamical systems as follows: Given the knowledge of an observable in the form of time series generated by experiment or simulation, what can we say about the evolution of the state? The Koopman operator theory provides a solution framework by describing the precise relationship between the evolution of observables and the evolution of state.

Consider the continuous-time dynamical system given in (0.2). Let \( g : S \to \mathbb{R} \) be a real-valued observable on this dynamical system. The collection of all such observables forms a linear vector space. The Koopman operator, denoted by \( U \), is a linear transformation on this vector space given by

\[
Ug(x) = g \circ T(x),
\]

(0.5)

where \( \circ \) denotes the composition operation. The linearity of the Koopman operator follows from the linearity of the composition operation, i.e.,

\[
U[g_1 + g_2](x) = [g_1 + g_2] \circ T(x) = g_1 \circ T(x) + g_2 \circ T(x) = U g_1(x) + U g_2(x).
\]

(0.6)

For continuous-time dynamical systems, we can define a one-parameter semi-group of Koopman operators, denoted by \( \{U^t\}_{t \geq 0} \), where each element of this semi-group is given by

\[
U^t g(x) = g \circ F^t(x),
\]

(0.7)

where \( F^t(x) \) is the flow map defined in (0.3). The linearity of \( U^t \) follows in the same way as the discrete-time case. The semi-group property of \( \{U^t\}_{t \geq 0} \) follows from the
Figure 0.1: Koopman viewpoint lifts the dynamics from state space to the observable space, where the dynamics is linear but infinite dimensional.

semi-group property of the flow map for autonomous dynamical systems given in (0.4),

\[ U^t U^s g(x) = U^t g \circ F^s(x) = g \circ F^t \circ F^s(x) = g \circ F^{t+s}(x) = U^{t+s} g(x). \quad (0.8) \]

An schematic representation of the Koopman operator is shown in figure 0.2. We can think of the Koopman operator viewpoint as a lifting of the dynamics from the state space to the space of observables. The advantage of this lifting is that it provides a linear rule of evolution - given by Koopman operator - while the disadvantage is that the space of observables is infinite dimensional. In the next section, we discuss the spectral theory of the Koopman operator which leads to linear expansions for data generated by nonlinear dynamical systems.
0.3 Koopman linear expansion

A somewhat naive but useful way of thinking about linear operators is to imagine them as infinite-dimensional matrices. Then, just like matrices, it is always useful to look at their eigenvalues and eigenvectors since they give a better understanding of how a matrix acts on a vector space. Let \( \phi_j : S \to \mathbb{C} \) be a complex-valued observable of the dynamical system in (0.1) and \( \lambda_j \) a complex number. We call the couple \((\phi_j, \lambda_j)\) an eigenfunction-eigenvalue pair of the Koopman operator if they satisfy

\[
U^t \phi_j = e^{\lambda_j t} \phi_j. \tag{0.9}
\]

An interesting property of the Koopman eigenfunctions, that we will use later, is that if \((\phi_i, \lambda_i)\) and \((\phi_j, \lambda_j)\) are eigenfunction-eigenvalue pairs, so is \((\phi_i \cdot \phi_j, \lambda_i + \lambda_j)\), that is

\[
U^t (\phi_i \cdot \phi_j) = (\phi_i \cdot \phi_j) \circ F^t = (\phi_i \circ F^t) \cdot (\phi_j \circ F^t) = U^t \phi_i \cdot U^t \phi_j = e^{(\lambda_i + \lambda_j) t} \phi_i \cdot \phi_j. \tag{0.10}
\]

Let us assume for now that all the observables of the dynamical system lie in the linear span of such Koopman eigenfunctions, that is,

\[
g(x) = \sum_{k=0}^{\infty} g_k \phi_k(x), \tag{0.11}
\]

where \( g_j \)'s are coefficients of expansion. Then we can describe the evolution of observables as

\[
U^t g(x) = \sum_{k=0}^{\infty} g_k e^{\lambda_k t} \phi_k(x), \tag{0.12}
\]

which says that the evolution of \( g \) has a linear expansion in terms of Koopman eigenfunc-
tions. If we fix the initial state $x = x_0$, we see that the signal generated by measuring $g$ over a trajectory, which is given by $U^t g(x_0) = g \circ F^t(x_0)$ is sum of (infinite number of) sinusoids and exponentials. This might sound a bit odd for nonlinear systems since sinusoids and exponentials are usually generated by linear systems.

It turns out that Koopman linear expansion in (0.12) holds for a large class of nonlinear systems, including the ones that have hyperbolic fixed points, limit cycles and tori. For these systems the spectrum of the Koopman operator consists of only eigenvalues and their associated eigenfunctions span the space of observables. Now we consider some of these systems in more detail. We borrow these examples from [4] where more details on the regularity of the system and related proofs can be found.

### 0.3.1 Examples of nonlinear systems with linear Koopman expansion:

1. Limit cycling is a nonlinear property in the sense that there is no linear system $(x = Ax)$ that can generate a limit cycle (i.e. isolated periodic orbit). If a limit cycle has time period $T$, then the signal generated by measuring $g(x)$ while $x$ is moving around the limit cycle is going to be $T$-periodic. From Fourier analysis, we have

   $$g(x(t)) = \sum_{k=0}^{\infty} g_j e^{ik(2\pi/T)t}.$$ 

   where $g_j$ are the Fourier coefficients. We can construct the eigenfunctions by letting $\phi_k(x(t)) = e^{ik(2\pi/T)t}$, and eigenvalues by $\lambda_k = ik(2\pi/T)$. It is easy to check that $(\phi_k, \lambda_k)$ satisfy (0.9), and the above equation is the Koopman linear expansion of $g$. 


2. Consider a nonlinear system with a hyperbolic fixed point, that is, the linearization around the fixed points yields a matrix whose eigenvalues don’t lie on the imaginary axis. There are a few well-known results in dynamical systems theory, such as Hartman-Grobman theorem [2], which state that the nonlinear system is conjugate to a linear system of the same dimension in a neighborhood of the fixed point. To be more precise, they say that there is an invertible coordinate transformation $y = h(x)$ such that the dynamics on $y$-coordinate is given by $\dot{y} = Ay$ (with the solution $y(t) = e^{At}y(0)$) and such that

$$F^t(x) = h^{-1}(e^{At}h(x))$$

i.e., to solve the nonlinear system, we can lift it to $y$-coordinate, and solve the linear system, and then transform it back to the $x$-coordinates. We first show the Koopman linear expansion for the linear systems, and then use the conjugacy to derive the expansion for the nonlinear system.

Let $\{v_j\}_{j=1}^n$ and $\{\lambda_j\}_{j=1}^n$ denote the eigenvectors and eigenvalues of $A$. The Koopman eigenfunctions for the linear system are simply the eigen-coordinates, that is

$$\tilde{\phi}_j(y) = <y, w_j>,$$

where $w_j$’s are normalized eigenvectors of $A^*$. To see this note that

$$U^t \tilde{\phi}_j(y) = <U^ty, w_j> = <e^{At}y, w_j> = <y, e^{At}w_j> = <y, e^{\lambda_j t}w_j> = e^{\lambda_j t} <y, w_j> = e^{\lambda_j t} \tilde{\phi}_j(y).$$
It is easy to show that $\phi_j(x) = \tilde{\phi}_j(h(x))$ are eigenfunctions of the Koopman operator for the nonlinear system. Other Koopman eigenfunctions can be easily constructed using the algebraic structure noted in (0.10).

To find the Koopman expansion for the nonlinear system it is easier to further transform $y$ into a decoupled linear system. If the matrix $A$ is diagonalizable and $V$ is the matrix of its eigenvectors, then the state variables of the diagonal system are, not surprisingly, the Koopman eigenfunctions, i.e.,

$$z = [z_1, z_2, \ldots, z_n]^T = V^{-1} y = [\tilde{\phi}_1(y), \tilde{\phi}_2(y), \ldots, \tilde{\phi}_n(y)]^T = [\phi_1(x), \phi_2(x), \ldots, \phi_n(x)]^T.$$

Now consider an observable of the nonlinear dynamical system $g(x) = g(h^{-1}(y)) = g(h^{-1}(Vz)) = \tilde{g}(z)$ where $\tilde{g}$ is real analytic in $z$ (and therefore $y$ as well). The Taylor expansion for of this observable in variable $z$ reads

$$g(x) = \tilde{g}(z) = \sum_{\{k_1, \ldots, k_n\} \in \mathbb{N}^n} \alpha_{k_1, \ldots, k_n} z_1^{k_1} z_2^{k_2} \cdots z_n^{k_n},$$

$$= \sum_{\{k_1, \ldots, k_n\} \in \mathbb{N}^n} \alpha_{k_1, \ldots, k_n} \phi_1^{k_1}(x) \phi_2^{k_2}(x) \cdots \phi_n^{k_n}(x),$$

Using the algebraic property of the Koopman eigenfunctions in (0.10), we can write the Koopman linear expansion of $g$ as

$$U^t g = \sum_{\{k_1, \ldots, k_n\} \in \mathbb{N}^n} \alpha_{k_1, \ldots, k_n} e^{(k_1 \lambda_1 + k_2 \lambda_2 + \cdots + k_n \lambda_n)t} \phi_1^{k_1} \phi_2^{k_2} \cdots \phi_n^{k_n}. $$

Recall that the original Hartman-Grobman theorem for nonlinear systems is local [2], in the sense that we knew the conjugacy exists for some neighborhood of
the fixed point. But the results in [4] has extended the conjugacy to the whole basin of attraction for stable fixed points using the properties of the Koopman eigenfunctions.

3. Now consider the motion in the basin of attraction of a (stable) limit cycle. The Koopman linear expansion for observables on such system can be constructed by, roughly speaking, combining the above two examples. That is, observables are decomposed into Koopman eigenfunctions, and each Koopman eigenfunction is a product of a periodic component, corresponding to the limit cycling, and a linearly contracting component for the stable motion toward the limit cycle. The development of this expansion is lengthy and can be found in [4].

The major class of dynamical systems for which the Koopman linear expansion does not hold is the class of chaotic dynamical systems. It turns out that for these systems, the eigenfunctions of the Koopman operator do not span the space of observables and we cannot decompose fluctuations of the system all into exponentials and sinusoids. In such cases the Koopman operator usually possesses a continuous spectrum. The continuous spectrum of the Koopman operator is similar to the power spectrum of a stationary stochastic process where the energy content is spread over a range of frequencies. In fact, if our dynamical system is measure-preserving (which is typically true for evolution on attractors) the spectral density of the Koopman operator coincides with the power spectral density of observable evolution. For more on this, and generally the connection between stochastic processes and Koopman representation of deterministic dynamics, see [5]. We also note that chaos in measure-preserving system is associated with continuous spectrum, but continuous spectrum can also be seen in non-chaotic systems. See the cautionary tale in [4]. The continuous spectrum is further discussed in [6, 7, 5].

What is more interesting is that some systems possess mixed spectra which is a
combination of eigenvalues and continuous spectrum. For these systems the evolution of a generic observable is composed of two parts: one part associated with eigenvalues and eigenfunctions which evolves linearly in time and a fully chaotic part corresponding to continuous spectrum. As such, the linear expansion (and the Koopman modes defined below) does hold for part of the data. Examples of systems with mixed spectra are given in [8, 9, 5].

0.4 Koopman Mode Decomposition (KMD)

A lot of times the data that is measured on a dynamical systems comes to us not from a single observable, but multiple observables. For example, when we are monitoring a power network system, we may have access to the time series of power generation and consumption on several nodes, or in the study of climate dynamics there are recordings of atmospheric temperature measured at different stations around the globe. We can easily integrate these multiplicity of time-series data into the Koopman operator framework and Koopman linear expansion.

We use $g : S \rightarrow \mathbb{R}^m$ to denote a vector-valued observable, i.e.,

$$ g = \begin{bmatrix} g^1 \\ g^2 \\ \vdots \\ g^m \end{bmatrix}, \quad g^j : S \rightarrow \mathbb{R}, \ 1 \leq j \leq m. $$

If we apply the linear Koopman expansion (0.12) to each $g^j$, we can collect all those
expansions into a vector-valued linear expansion for $g$,

$$U^t g(x) = \sum_{k=0}^{\infty} g_k e^{\lambda_k t} \phi_k(x). \quad (0.13)$$

The above expansion is the **Koopman Mode Decomposition (KMD)** of observable $g$ and $g_k$ is called the **Koopman mode** of observable $g$ at the eigenvalue $\lambda_k$. Koopman modes are in fact the projection of observable onto the Koopman eigenfunctions. We can think of $g_k$ as a structure (or shape) within the data that evolves as $e^{\lambda_k t}$ with time. Let us examine the concept of the Koopman modes in the examples mentioned above. In the context of power networks, we can associate the network instabilities with the Koopman eigenvalues that grow in time, that is $\lambda_k > 0$, and as such, the entries of Koopman mode $g_k$ give the relative amplitude of each node in unstable growth and hence predict which nodes are most susceptible to breakdown. In the example of climate time series, the Koopman modes of temperature recordings give us the spatial pattern (depending on the location of stations) of temperature change that is proportional to $e^{\lambda_k t}$, and therefore indicate the spots with extreme variations.

In some physical problems, we have a **field of observables**, i.e., an observable that assigns a physical field to each element of the state space. A prominent example, that we focus on in the next chapter, is a fluid flow. The pressure field over a subdomain of the flow, or the whole vorticity field, are two examples of field of observable defined on a flow, since the knowledge of the flow state (e.g. instantaneous velocity field) uniquely determines those fields. We can formalize the notion of a field of observable as a function $g : (S, \Omega) \rightarrow \mathbb{R}$ where $\Omega$ is the flow domain and $g(x, z)$ determines the value of the field at point $z$ in the flow domain when the flow is at state $x$. The Koopman linear expansion

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Figure 0.2: Koopman Mode Decomposition fully describes the evolution of observables on systems with Koopman discrete spectrum, but not for chaotic systems which have continuous spectrum.

for \( g \) would be

\[
U^t g(x, z) = \sum_{k=0}^{\infty} g_k(z) e^{\lambda_k t} \phi_k(x),
\]

(0.14)

where the Koopman mode \( g_k(z) \) is a fixed field by itself, and similar to the Koopman mode vectors, determines a shape function on \( \Omega \) which grows with the amplitude \( e^{\lambda_k t} \) in time. In a fluid flow, the Koopman modes of vorticity, are steady vorticity fields, and the whole flow can be decomposed into such fields, with amplitudes that grow as \( e^{\lambda_k t} \).

0.5 History of Koopman operator theory and its application to data analysis

The Koopman operator formalism originated in the early work of Bernard Koopman in 1931 [10]. He introduced the linear transformation that we now call the Koopman operator, and realized that this transformation is unitary for Hamiltonian dynamical systems (the “\( U \)” notation comes from unitary property). This observation by Koopman
inspired John Von Neumann to give the first proof for a precise formulation of ergodic hypotheses, known as \textit{mean ergodic theorem} \cite{11}. In the next year, they wrote a paper together, in which they introduced the notion of the \textit{spectrum of a dynamical system}, i.e. the spectrum of the associated Koopman operator, and noted the connection between chaotic behavior and the continuous part of the Koopman spectrum \cite{12}.

For several decades after the work of Koopman and Von Neumann, the notion of Koopman operator was mostly limited to the study of measure-preserving systems; you could find it as the unitary operator in the proof the mean ergodic theorem or discussions on the spectrum of measure-preserving dynamical systems \cite{13, 14}. It seldom appeared in other applied fields until it was brought back to the general scene of dynamical system by two articles in 2004 and 2005 \cite{9, 6}. The first paper showed how we can construct important objects like the invariant sets in high-dimensional state spaces from data. It also emphasized the role of nontrivial eigenvalues of the Koopman operators to detect the periodic trends of dynamics amidst chaotic data. The second paper discussed the spectral properties of the Koopman operator further, and introduced the notion of Koopman modes. Both papers also discussed the idea of applying Koopman methodology to capture the regular components of data in systems with combination of chaotic and regular behavior.

In 2009, the idea of Koopman modes was applied to a complex fluid flow, namely, a jet in a cross flow \cite{15}. This work showed the promise of KMD in capturing the dynamically relevant structures in the flow and their associated time scales. Unlike other decomposition techniques in flows, KMD combines two advantageous properties: it makes a clear connection between the measurements in the physical domain and the dynamics of state space (unlike proper orthogonal decomposition), and it is completely data-driven (unlike the global mode analysis). The work in \cite{15} also showed that KMD can be computed through a numerical decomposition technique known as \textit{Dynamic Mode}
Decomposition (DMD) [16]. Since then, KMD and DMD has become immensely popular in analyzing the nonlinear flows [16, 17, 18, 19, 20, 21, 22, 23, 24]. A review of the Koopman theory in the context of flows can be found in [25].

In the recent years, the extent of KMD applications for data-driven analysis has enormously grown. Some of these applications include model reduction and fault detection in energy systems for buildings [26, 27], coherency identification and stability assessment in power networks [28, 29], extracting spatio-temporal patterns of brain activity [30], background detection and object tracking in videos [31, 32] and design of algorithmic trade strategies in finance [33].

Parallel to the applications, the computation of Koopman spectral properties (modes, eigenfunctions and eigenvalues) has also seen a lot of major advancements. For post-transient systems, the Koopman eigenvalues lie on the unit circle and Fourier analysis techniques can be used to find the Koopman spectrum and modes [9, 5]. For dissipative systems, the Koopman spectral properties can be computed using a theoretical algorithm known as Generalized Laplace Analysis [34, 35]. In practice however, DMD remains the more popular technique for computation of Koopman spectrum from data. In [36], the idea of Extended DMD was introduced for general computation of Koopman spectrum by sampling the state space and using a dictionary of observables. The works in [37] and [38] discussed the linear algebraic properties of the algorithm and suggested new variations for better performance and wider applications. New variants of DMD were also introduced in [39] to unravel multi-time-scale phenomena and in [40] to account for linear input to the dynamical system. Due to constant growth in the size of the available data, new alterations or improvements of DMD are also devised to handle larger data sets [41, 42], different sampling techniques [43, 38, 42] and noise [44, 45]. The convergence of DMD-type algorithms for computation of Koopman spectrum was established in [46] and [47].
The ultimate goal of many data analysis techniques is to provide information that can be used to predict and manipulate a system to our benefit. Application of the Koopman operator techniques to data-driven prediction and control are just being developed, with a few-year lag behind the above work. This lag is perhaps due to the need to account for the effect of input in the formalism, but promising results have already appeared in this line of research. The work in [48] showed an example of optimal controller which was designed based on a finite-dimensional Koopman linear expansion of nonlinear dynamics. The works in [49, 50] have developed a framework to build state estimators for nonlinear systems based on Koopman expansions. More recent works, have shown successful examples of Koopman linear predictors for nonlinear systems [51], and optimal controllers of Hamiltonian systems designed based on Koopman eigenfunctions [52]. More recent applications include feedback control of fluid flows via using Koopman linear models computed from data in a model-predictive control framework [53, 54, 55].
Bibliography


