"Go forth and multiply." - Genesis 24.2
(Biblical proof that you should not forget your new math skills)

1) \textit{Mitragotri et al., Biophysical Journal, 2006} \ [30 pts]

A spherical cell of radius \( r=1 \) has a nucleus at the center of the cell with radius \( r=\varepsilon \). Spokes extend from the nucleus outward through the cytoplasm as shown in figure 1. This problem illustrates how molecules that bind to these spokes and diffuse along the fibers become concentrated near the nucleus.

A. \ [10 pts] a single molecule diffusing freely in the cytoplasm has a uniform (i.e. constant) probability of being found anywhere in the cytoplasm. Derive the probability density \( \rho(r) \) such that the probability of finding the particle between \( r \) and \( r+dr \) from the center of the cell is \( \rho(r)dr \). Hints: Shell balance. For \( r < \varepsilon \) and \( r > 1 \), \( \rho(r) = 0 \). Also, the probability of finding the molecule between \( \varepsilon \) and \( r_0 \) is

\[
P[\varepsilon < r < r_0] = \int_{\varepsilon}^{r_0} \rho(r)dr.
\]

B. \ [20 pts] At time \( t = 0 \), the molecule binds to one of the spokes radiating from the nucleus and begins to diffuse along the spoke according to a one-dimensional diffusion equation. The position at which it binds is related to the distribution you found in part A, leading to the initial boundary value problem below.

\[
PDE: \quad \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} \quad \varepsilon < r < 1, \quad t > 0
\]

\[
IC: \quad u(r, 0) = \rho(r)
\]

\[
BC: \quad \left. \frac{\partial u}{\partial r} \right|_{r=\varepsilon} = \left. \frac{\partial u}{\partial r} \right|_{r=1} = 0
\]

You know a few strategies that will work for this problem. Solve it and sketch the functions \( \rho(r)/(4\pi r^2) \) and \( u(r, t = \infty)/(4\pi r^2) \) as functions of \( r \).
2) “Classical paths” and quantum density matrices [16 pts]

The “density matrix” \( \langle x_2 | e^{\beta H} | x_1 \rangle \) where \( \beta H \) is the Hamiltonian in units of \( kT \) is used to compute thermally averaged properties in quantum mechanics. For example the density matrix is useful for proton and electron transfer rate constants. The density matrix calculation is a sum over all paths with the constrained endpoints

\[
x(0) = x_1 \quad \text{and} \quad x(\beta h) = x_2
\]

and with each path weighted by

\[
\exp\{-\phi[x(t)]\} \quad \text{where} \quad \phi[x(t)] = \frac{1}{\hbar} \int_0^{\beta h} dt \left( \frac{1}{2} m x'(t)^2 + V(x(t)) \right)
\]

The path with the largest weight in the sum is the path that minimizes the functional \( \phi[x(t)] \). Find the equations of motion for the path with maximum weight. Why is the path interpreted as a classical path on an inverted potential?

3) Convection-diffusion problem [24 pts]

A. [12 pts] Solve the pure-convection equation:

\[
PDE \quad U_t + V U_x = 0 \\
IC \quad U(x, 0) = \delta[x]
\]

to obtain the new coordinates \((\tau, z)\) where \( \tau = t \) and \( z = x - Vt \). Outline your steps clearly because we gave you the solution!

B. [12 pts] Solve the convection-diffusion equation

\[
PDE \quad U_t = D U_{xx} - V U_x \\
IC \quad U(x, 0) = \delta[x]
\]

Start by changing from \((x, t)\) coordinates to the \((z, \tau)\) coordinates from part A.
4) Short answer questions [30 + 2pts]

A. [6 pts] If w(x,t) is the solution to problem 2B, how would you solve the problem below?

\[ \begin{align*}
    PDE & \quad U_t = DU_{xx} - VU_x \\
    IC & \quad U(x, 0) = f(x)
\end{align*} \]

B. [6 pts] If w(x,t) is the solution to problem 2B, how would you solve the problem below?

\[ \begin{align*}
    PDE & \quad U_t = DU_{xx} - VU_x + \delta[x]f(t) \\
    IC & \quad U(x, 0) = 0
\end{align*} \]

C. [4 pts] We obtained similarity solutions for PDEs in terms of a dimensionless similarity variable of the form \( \eta = x/g(t) \) where \( x \) was a distance. What two things are necessary for a similarity solution to exist?
D. [4 pts] Suppose the normalized eigenfunctions of a Hermitian operator are \( u_k(x) \) where \( k = 1, 2, 3, \ldots \) but that we only know one of them, say \( u_1(x) \). Then suppose that you take an arbitrary function \( f(x) \) and use the Gram-Schmidt procedure to make a new function \( \phi(x) \) that is orthogonal to \( u_1(x) \),

\[
\phi(x) = f(x) - \langle u_1(x) | f(x) \rangle u_1(x)
\]

Is \( \phi(x) \) an eigenfunction of the Hermitian operator? Explain why or why not?

E. [12 pts] We learned that a conformal mapping “\( w = g(z) \)” where \( g \) is an analytic function of the complex number \( z = x + iy \) and \( w = u(x,y) + iv(x,y) \) preserves Laplace’s equation in two dimensions, i.e.

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \quad \rightarrow \quad \frac{\partial^2 \Phi}{\partial u^2} + \frac{\partial^2 \Phi}{\partial v^2} = 0
\]

Laplace’s equation is preserved because the Cauchy-Reimann equations hold for analytic functions (i.e. maps like \( g(z) \)) and these equations require that

\[
\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}
\]

Change variables from \((x,y)\) to \((u,v)\) for Poisson’s equation in the case of “uniform heat generation”

\[
- \left[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} \right] = 1
\]

The Cauchy-Reimann equation still apply and they will help you eliminate many terms. **Outline a strategy** (in about three lines of detail) to solve the uniform heat generation problem on the region \( R \) with boundary conditions \( \Phi = 0 \) on the boundary of \( R \). Assume you know the mapping \( z = g(w) \) to the region \( R' \) and the inverse mapping \( z = g^{-1}(w) \).
Table 6.1 Laplace Transforms

<table>
<thead>
<tr>
<th>u(t)</th>
<th>U(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>s⁻¹</td>
</tr>
<tr>
<td>eᵃᵗ</td>
<td>s⁻¹⁻ᵃ</td>
</tr>
<tr>
<td>tⁿ, n a positive integer</td>
<td>sⁿ⁻¹⁻ⁿ</td>
</tr>
<tr>
<td>sin at and cos at</td>
<td>s²ᵃ⁻ᵃ²</td>
</tr>
<tr>
<td>sinh at and cosh at</td>
<td>s²⁻ᵃ⁻²</td>
</tr>
<tr>
<td>eᵃᵗ sin bt</td>
<td>(s⁻ᵃ⁻ᵇ²)⁻¹</td>
</tr>
<tr>
<td>eᵃᵗ cos bt</td>
<td>(s⁻ᵃ⁻ᵇ²)⁻¹</td>
</tr>
<tr>
<td>tⁿ exp(at)</td>
<td>sⁿ⁻¹⁻ⁿ⁻¹</td>
</tr>
</tbody>
</table>
| H(t-a)      | s⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻¹⁻�
Fundamental point source solutions:

1D: \( u(x) = \frac{-|x-x_0|}{2} \quad x_0 = \text{location of source} \)

2D: \( u(r) = \frac{\ln(1/r)}{2\pi} \quad r = \text{distance from source} \)

3D: \( u(r) = \frac{1}{4\pi r} \quad r = \text{distance from source} \)
\( p[e < r < r_0] = \int_e^{r_0} \rho(r) \, dr = \int_e^{r_0} 4\pi r^2 c \, dr = \frac{4\pi}{3} (r_0^3 - e^3) c \)

\( p[e < r < 1] = 1 \)

\[ \Rightarrow \quad c = \left[ \frac{4\pi}{3} (1 - e^3) \right]^{-1} \]

\[ \Rightarrow \quad \rho(r) = \frac{4\pi r^2}{\frac{4\pi}{3} (1 - e^3)} = \frac{3r^2}{1 - e^3} = \rho(r) \]

\( \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial r^2} \)

\( u(r, 0) = \rho(r) \)

\( \frac{\partial u}{\partial r} \bigg|_{r = e} = \frac{\partial u}{\partial r} \bigg|_{r = 1} = 0 \)

Separate variables → eigenfunctions \( \frac{dR}{dr} = \frac{d^2}{dr^2} \)

\[ \frac{T}{T_1} = -\mu^2 = \frac{R''}{R} \]

\[ R = A \sin(\mu r + B) \cos \mu r \]

\[ R = A \cos(\mu r + B) \]

\[ R'(r) = -\mu A \sin(\mu r + B) \]

\[ R'(1) = -\mu A \sin(\mu + B) = 0 \]

\( -\frac{d^2}{dr^2} \cos(\mu r + B) \)

\[ -\frac{1}{r^2} \frac{d}{dr} \left[ r^2 \frac{d}{dr} \right] \theta(r) \theta(1) \]

\[ -\frac{1}{r^2} \frac{d}{dr} \left[ r^2 \frac{d}{dr} \right] \theta(1) \theta(1) \]
\[ \Rightarrow \mu + B - \left( \mu \epsilon + B \right) = k \pi \quad k = 0, 1, 2, \ldots \]

so \( \mu (1 - \epsilon) = k \pi \)

and \( \mu + B = \pi \epsilon \)

take \( n = 0 \) \( \Rightarrow B = -\mu \epsilon \)

and \( \mu = \frac{k \pi}{1 - \epsilon} \)

so \( R_k(r) = \cos \left( \frac{k \pi r}{l - \epsilon} \right) \)

| \( R_k(r) = \cos \left( \frac{\pi k (r - \epsilon)}{l - \epsilon} \right) \) |

| Eigenfunction |

\[ \text{solution:} \quad u(r, t) = \sum_{k=0}^{\infty} c_k \epsilon^{-\frac{(k \pi)^2}{1 - \epsilon}} \cos \left( \frac{\pi k (r - \epsilon)}{l - \epsilon} \right) \]

use orthogonality to find \( c_k \)’s.

\[ u(r, 0) = \sum_{k=0}^{\infty} c_k \cos \left( \frac{\pi k (r - \epsilon)}{l - \epsilon} \right) \]

\[ \int_{\epsilon}^{1} R_m(r) u(r, 0) \, dr = \sum_{k=0}^{\infty} c_k \int_{\epsilon}^{1} R_m(r) R_k(r) \, dr = c_m \| R_m \| ^2 \]

[Diagram]

\[ \Rightarrow c_k = \frac{\int_{\epsilon}^{1} R_k(r) \rho(r) \, dr}{\| R_k(r) \| ^2} \]

\[ u(r, t) = \sum_{k=0}^{\infty} c_k \epsilon^{-\frac{(k \pi)^2}{1 - \epsilon}} \cos \left( \frac{\pi k (r - \epsilon)}{l - \epsilon} \right) \]
\[ \delta \phi = \frac{i}{\hbar} \int_0^{\beta \hbar} dt \left[ \frac{1}{2} m (x' + \delta x')^2 + V(x + \delta x) \right] - \frac{1}{\hbar} \int_0^{\beta \hbar} dt \left[ \frac{1}{2} m x'^2 + V(x) \right] + \frac{1}{\hbar} \int_0^{\beta \hbar} \frac{2V}{dx} \delta x dt \]

\[ = \frac{1}{\hbar} \int_0^{\beta \hbar} dt \left\{ \frac{d}{dt} \left( \sqrt{m} x' \delta x' + \delta x'^2 \right) + \frac{1}{\hbar} \int_0^{\beta \hbar} \frac{2V}{dx} \delta x dt \right\} \]

\[ = \frac{1}{\hbar} \int_0^{\beta \hbar} \delta x(t) \left\{ \frac{d}{dx} \left[ \frac{2V}{dx} - \left[ m x' \right]' \right] \right\} dt \]

\[ \frac{\delta \phi}{\delta x(t)} = \frac{d}{dx} \left[ \frac{2V}{dx} - m \frac{d^2}{dt^2} \right]_{x(t)} \]

The path of maximum weight has

\[ \frac{\delta \phi}{\delta x(t)} = 0 \quad \forall t, \quad \Rightarrow \quad \frac{d}{dx} \frac{2V}{dx} = m \frac{d^2 x}{dt^2} \]

Like \( F = ma \) but \( V \to (-V) \).

is on an "upside-down" potential energy surface.
3. A method of characteristics

We want LHS of $U_t + VU_x = 0$ to look like $\frac{dU}{dt}$

$$\frac{du}{dt} = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} dx$$

3. $\frac{du}{dt} = \frac{\partial u}{\partial t} dt + \frac{\partial u}{\partial x} \frac{dx}{dt} = U_t + VU_x$

Correspondence of $t=0$

$$s(x,0) = x, \quad z(x,0) = 0$$

$$\begin{align*}
\frac{dt}{dz} &= 1 \quad \rightarrow \quad t = z + c_t(s) \\
\frac{dx}{dz} &= V \quad \rightarrow \quad x = Vz + c_x(s)
\end{align*}$$

$$\frac{du}{dz} = 0 \rightarrow u = f(s) \quad \text{const. of integration}$$

$$u(x,t) = f(x-Vt)$$

$\downarrow$ ICs,

$$u(x,0) = \delta[x] \rightarrow \quad u(x_t) = \delta[x-Vt]$$

3B

$$\begin{align*}
\frac{\partial u}{\partial t} &= \frac{\partial u}{\partial t} \frac{\partial z}{\partial t} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial t} \quad \Rightarrow \quad U_t = U_t - VU_z \\
\frac{\partial u}{\partial x} &= \frac{\partial u}{\partial x} \frac{\partial z}{\partial x} + \frac{\partial u}{\partial z} \frac{\partial z}{\partial x} \quad \Rightarrow \quad U_x = U_x \\
\frac{\partial^2 u}{\partial x^2} &= \frac{\partial}{\partial x} \left[ U_z \right] = \frac{\partial u_x}{\partial z} \frac{\partial z}{\partial x} + \frac{\partial u_z}{\partial z} \frac{\partial z}{\partial x} \quad \Rightarrow \quad U_{xx} = U_{zz}
\end{align*}$$

PDE: (use formulas above)

$u_t = \nabla u_{xx} - VU_x$ \quad $\Rightarrow \quad u_t - VU_z = \nabla u_{zz} - VU_z$

$\Rightarrow \quad u_t = \nabla u_{zz}$ \quad PDE
\[ x \to \pm \infty, \ z \to \pm \infty \quad u = 0 \quad u(\pm \infty, t) = 0 \]

**ICs**

\[ u(x,0) = \delta(x) \quad z(x,t) = x-Vt \]

\[ \Rightarrow u(z,0) = \delta(z) \]

**BCs**

\[ u_z = \partial_z u \]

\[ u(0,0) = \delta(0) \]

**Fourier Transform**

\[ \hat{u}(\omega,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(z) e^{-i\omega z} dz \]

\[ \hat{u}_{\omega} = -i \omega^2 \hat{u} \]

\[ \Rightarrow \hat{u} = \hat{u}(\omega,0) e^{-i\omega^2 t} \]

\[ \hat{u} = \frac{1}{\sqrt{2\pi}} e^{-i\omega^2 t} \]

**Fourier Transform of ICs**

\[ \hat{u}(\omega,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \delta(z) e^{-i\omega z} dz = \frac{1}{\sqrt{2\pi}} \]

**ICs**

\[ u(z,t) = \frac{a}{\sqrt{\pi}} e^{-a^2 z^2} \]

\[ = \frac{1}{2\sqrt{\pi} \beta t} e^{-\frac{(x-Vt)^2}{4\beta t}} = u(x,t) \]
transient response Green's function (impulse response) is an infinite convolution integral

\[ u(x, t) = \int_{-\infty}^{\infty} w(x - x_0, t) f(x_0) \, dx_0 \]  

now we have a time dependent input. "w" is solution to

\[ u_t = \beta u_{xx} - \nu u_x + \delta(x) \delta(t) \]

\[ u(x, t) = \int_{0}^{t} w(x, t - t_0) f(t_0) \, dt_0 \]  

IC must collapse onto one of the BCs.
PDE must collapse to an ODE in \( y \) with no explicit \( x \) or \( t \) dependence.

\( \phi(x) \) is \( \perp \) to \( u_1(x) \) but it still overlaps with \( u_2(x), u_3(x), \ldots \) there is no reason to expect that \( \phi(x) \) is an eigenvector
Best strategy

1. Use $w = z^2$ 1st but be careful! $\mu_3$ is not $\nabla^2 \bar{\Phi} = 0$. Use chain rule

\[
\nabla_{\bar{w}} \bar{\Phi} \quad \nabla_{\bar{w}} \bar{\Phi} = f(\bar{w})
\]

2. Use fundamental 2D point source solution: (ignores BC's)

\[
\bar{\Phi}_1(u,v) = -\frac{1}{2\pi} \int_{R} d\bar{w}_0 \frac{f(\bar{w}_0)}{g(\bar{w}_0)} \ln |\bar{w} - \bar{w}_0| \
R = \begin{array}{c}
A \\
B \\
do\bar{w}_0 = r_0 d\rho_0 d\theta_0
\end{array}
\]

3. Solve

\[
\nabla_{uv}^2 \bar{\Phi}_2 = 0 \quad \bar{\Phi}_2(u,v) = \bar{\Phi}_1(u,v)
\]

on boundary of $R$

4. Use superposition to further subdivide problem:

\[
\bar{\Phi}_2 = \bar{\Phi}_{2A} + \bar{\Phi}_{2B}
\]

\[
\nabla_{uv}^2 \bar{\Phi}_{2A} = 0 \quad \bar{\Phi}_{2A} = \begin{cases} O & x \in B \\
\bar{\Phi}_1 & x \in A \\
\end{cases}
\]

\[
\nabla_{uv}^2 \bar{\Phi}_{2B} = 0 \quad \bar{\Phi}_{2B} = \begin{cases} O & x \in B \\
\bar{\Phi}_1 & x \in A \\
\end{cases}
\]

now a straightforward annulus problem.

5. Those are the "inverse mapping"

\[
\bar{\Phi}(\bar{z}) = \bar{\Phi}_1(\bar{w}(\bar{z})) + \bar{\Phi}_{2A}(\bar{w}(\bar{z})) - \bar{\Phi}_{2B}(\bar{w}(\bar{z}))
\]
Euler-Lagrange
\[ \frac{\delta F}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx} \left[ \frac{\partial f}{\partial y'} \right] \]

Method of characteristics
\[ \zeta(x,0) = 0 \]
\[ \xi(x,0) = x \]
choose \( d\zeta/d\tau , dx/d\tau \)
so \( u_t + u(x,t)u_x = f(u) \)
becomes \( du/d\tau = f(u) \)
\[ \frac{du}{d\tau} = u_t \frac{dt}{d\tau} + u_x \frac{dx}{d\tau} = f(u) \]

Leibniz' rule
\[ I(x) = \int g_z(x) f(x,y) dy \]
\[ \frac{dT}{dx} = g_z'(x) f(x,g_z(x)) \]
\[ - g_i'(x) f(x,g_i(x)) \]
\[ + \int g_i(x) \frac{\partial f(x,y)}{\partial x} dy \]
eigenfunctions of common Sturm-Liouville operators
\[ - \frac{d^2}{dx^2} \cos(\mu x + \beta) \]
\[ - \frac{1}{x} \frac{d}{dx} \left( x \frac{d}{dx} \right) g_0(\mu x) , \gamma_0(\mu x) \]
\[ - \frac{1}{x^2} \frac{d}{dx} \left( x^2 \frac{d}{dx} \right) \]
Table 6.1 Laplace Transforms

<table>
<thead>
<tr>
<th>Function</th>
<th>Laplace Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(t)$</td>
<td>$U(s)$</td>
</tr>
<tr>
<td>$1$</td>
<td>$s^{-1}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$e^{at}$</td>
<td>$\frac{1}{s-a}$, $s &gt; a$</td>
</tr>
<tr>
<td>$t^n$, $n$ a positive integer</td>
<td>$\frac{n!}{s^{n+1}}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$\sin at$ and $\cos at$</td>
<td>$\frac{a}{s^2 + a^2}$ and $\frac{s}{s^2 + a^2}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$\sinh at$ and $\cosh at$</td>
<td>$\frac{a}{s^2 - a^2}$ and $\frac{s}{s^2 - a^2}$, $s &gt;</td>
</tr>
<tr>
<td>$e^{at} \sin bt$</td>
<td>$\frac{b}{(s-a)^2 + b^2}$, $s &gt; a$</td>
</tr>
<tr>
<td>$e^{at} \cos bt$</td>
<td>$\frac{s-a}{(s-a)^2 + b^2}$, $s &gt; a$</td>
</tr>
<tr>
<td>$t^n \exp(at)$</td>
<td>$\frac{n!}{(s-a)^{n+1}}$, $s &gt; a$</td>
</tr>
<tr>
<td>$H(t-a)$</td>
<td>$s^{-1} \exp(-as)$, $s &gt; 0$</td>
</tr>
<tr>
<td>$\delta(t-a)$</td>
<td>$\exp(-as)$</td>
</tr>
<tr>
<td>$H(t-a)f(t-a)$</td>
<td>$F(s) \exp(-as)$</td>
</tr>
<tr>
<td>$f(t)e^{-at}$</td>
<td>$F(s) + a$</td>
</tr>
<tr>
<td>$H(t-a)f(t)$</td>
<td>$e^{-as} \mathcal{L}(f(t+a))$</td>
</tr>
<tr>
<td>$\text{erf} \sqrt{t}$</td>
<td>$s^{-1}(1+s)^{-1/2}$, $s &gt; 0$</td>
</tr>
<tr>
<td>$\frac{1}{\sqrt{t}} \exp \left( \frac{-a^2}{4t} \right)$</td>
<td>$\sqrt{\pi/s} \exp(-a\sqrt{s})$, $(s &gt; 0)$</td>
</tr>
<tr>
<td>$1 - \text{erf} \left( \frac{a}{2\sqrt{t}} \right)$</td>
<td>$s^{-1} \exp(-a\sqrt{s})$, $s &gt; 0$</td>
</tr>
<tr>
<td>$\frac{a^2}{2t^{3/2}} \exp \left( \frac{-a^2}{4t} \right)$</td>
<td>$\sqrt{\pi} \exp(-a\sqrt{s})$, $s &gt; 0$</td>
</tr>
<tr>
<td>$u^{(n)}(t)$</td>
<td>$s^n U(s) - s^{n-1} u(0) - s^{n-2} u'(0) - \ldots - u^{(n-1)}(0)$</td>
</tr>
<tr>
<td>$\int_0^t u(\tau) v(t-\tau) , d\tau$</td>
<td>$U(s) , V(s)$</td>
</tr>
</tbody>
</table>

Table 6.2 Fourier Transforms

<table>
<thead>
<tr>
<th>Function</th>
<th>Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$u(x)$</td>
<td>$U(\xi)$</td>
</tr>
<tr>
<td>$e^{-ax^2}$</td>
<td>$\sqrt{\frac{\pi}{a}} e^{-\xi^2/4a}$</td>
</tr>
<tr>
<td>$H(x)$</td>
<td>$\pi \delta(\xi) - \frac{i}{\xi}$</td>
</tr>
<tr>
<td>$\delta(x-x_0)$</td>
<td>$e^{i\xi x_0}$</td>
</tr>
<tr>
<td>$e^{-a</td>
<td>x</td>
</tr>
<tr>
<td>$H(a-</td>
<td>x</td>
</tr>
<tr>
<td>$u^{(n)}(x)$</td>
<td>$(-i\xi)^n U(\xi)$</td>
</tr>
<tr>
<td>$u * v$</td>
<td>$F(\xi) , G(\xi)$</td>
</tr>
</tbody>
</table>

Euler–Lagrange: $F[y] = \int f \, dx$

\[ \frac{\delta F}{\delta y(x)} = \frac{\partial f}{\partial y} - \frac{d}{dx} \left[ \frac{\partial f}{\partial y'} \right] \]

Method of characteristics: make PDE

in $(x,t) \rightarrow$ ODE in $[\tau; s(\text{parameter})]$

i.e. seek $\frac{d}{d\tau} = \frac{dt}{d\tau} \frac{dx}{dt} + \frac{du}{d\tau} = f(u)$

and set $\tau(x,0) = 0$ and $s(x,0) = 1$. 