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Topics in Optimal Distributed Control

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by

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E. Jensen, B. Bamieh, and J. P. Epperlein. "Localization of the LQR Feedback Kernel in Spatially-Invariant Problems over Sobolev Spaces." (To be published, 2020 IEEE 59th Conference on Decision and Control (CDC)).

E. Jensen and B. Bamieh. "An Explicit Parametrization of Closed Loops for Spatially Distributed Controllers with Sparsity Constraints." (To be published, IEEE Transactions on Automatic Control).

E. Jensen and B. Bamieh. "On the gap between system level synthesis and structured controller design: the case of relative feedback." 2020 Annual American Control Conference (ACC). IEEE, 2020.

E. Jensen and B. Bamieh. "A backstepping approach to system level synthesis for spatially invariant systems." 2020 Annual American Control Conference (ACC). IEEE, 2020.

E. Jensen and B. Bamieh. "Optimal spatially-invariant controllers with locality constraints: A system level approach." 2018 Annual American Control Conference (ACC). IEEE, 2018.

E. Jensen and J. R. Marden. "Optimal utility design in convex distributed welfare games." 2018 Annual American Control Conference (ACC). IEEE, 2018.

J. C. Doyle, Y. Nakahira, Y. P. Leong, **E. Jensen** et al, "Teaching control theory in high school." 2016 IEEE 55th Conference on Decision and Control (CDC). IEEE, 2016.

Abstract

Topics in Optimal Distributed Control

by

Emily Jensen

We consider the optimal controller design problem for linear time-invariant, spatially-distributed systems. The controller to be designed is itself a distributed system; each subcontroller component is restricted to have access to only a local subset of system information, which is shared across the network according to an underlying communication graph. The design problem of interest is to synthesize optimal controllers (with respect to some performance measure) subject to this limited information sharing architecture. In this dissertation, we contribute to two directions of research in this setting: i) analysis of *constraints* that ensure such localization and can be imposed in a tractable manner, and ii) characterization of settings in which the *unconstrained* centralized optimal controller has an inherent degree of spatial localization.

For direction (i), we follow the 'System Level Synthesis' (SLS) approach and consider directly designing the *closed-loops* as opposed to the controller or corresponding Youla parameter. Structural constraints on the closed-loop can be imposed in a convex manner, and we demonstrate that the optimal controller design problem subject to closed-loop transfer function sparsity constraints is a convex relaxation of the optimal controller design problem subject to structural constraints on a controller state-space realization (implementation). We provide an implicit parameterization of all achievable closed-loop mappings for a broad class of systems, including continuous- or discrete-time spatially-invariant systems over an infinite spatial domain. Under certain assumptions, in the state feedback setting we convert this implicit closed-loop parameterization to an explicit affine linear parameterization.

Our parameterizations allow for conversion of the closed-loop structured $\mathcal{H}_2$ optimal controller design problem to a standard model matching problem with finitely many transfer function parameters, allowing for analytic solutions in certain problem settings. We further take a step toward quantifying the performance gap between structured closed-loop transfer function design and structured controller realization design by studying the setting of relative feedback controllers. To do so, we provide a compact and convex characterization of all relative feedback controllers, and demonstrate that the relative feedback requirement can be imposed as a convex constraint on the closed-loop in certain problem settings. We use this characterization to show that the optimal relative feedback controller design problem subject to closed-loop structural constraints may be infeasible.

In direction (ii), we consider the optimal control of PDEs over a Sobolev space. We demonstrate that the optimal state feedback is a spatial convolution operator given by an exponentially decaying convolution kernel, thus enabling implementation with a localized

architecture, extending previous results in the L_2 setting. The main tool we utilize is a transformation from a Sobolev to an L_2 space, which is constructed from a spectral factorization of the spatial frequency weighting matrix of the Sobolev norm.

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Chapter 1

Introduction

The field of optimal control has historically focused on the control of a single system. More recently though, systems to be controlled are often *spatially-distributed*, composed of multiple interacting subsystem components. Relevant examples include satellite constellations [1], flight formations [2], vehicular platoons [3], the power grid, and many others. In this spatially-distributed setting, the controller to be designed is itself a distributed system, and there are typically additional requirements of controller locality which encode constraints about which site measurements the control signal for each site can depend on. Throughout this dissertation, the design problem of interest is to synthesize optimal controllers (with respect to some performance measure) that account for the structural constraints defining this limited information sharing architecture. In particular, we consider two lines of research in this setting: (i) analysis of methods to explicitly enforce this structure in a constrained optimal control problem, and (ii) characterization of problem settings for which the unconstrained (centralized) policy is inherently localized.

Many recent works have highlighted the underlying challenges of the optimal controller design problem subject to structural constraints, and highlighted its underlying challenges. For example, [4, 5] have used ℓ^1 regularization to promote sparsity in controller design, and [6] has considered implementing optimal centralized control policies with structured distributed approximations. A widely studied approach is to impose constraints directly on the input-output mapping that defines the controller, such as imposing a sparsity pattern on the controller transfer matrix. It is well-known however, that structural constraints on the controller transfer matrix, such as a prescribed sparsity pattern, are non-convex expect in special cases, e.g. quadratically invariant [7] and funnel causal [8] systems. In addition, sparsity constraints on the controller transfer matrix may not actually be the 'best' way to account for a given communication structure.

Indeed, recent works have begun to emphasize the practical importance of looking at the structure of *implementations* of the controller transfer function [9–11] rather than the transfer function itself (e.g. structure of state-space realizations), and this dissertation contributes to this line of work. Characterization of the set of controllers which have structured realizations that are stabilizable and detectable remains an open problem, although recent works have provided preliminary results [12–14].

In part (i) of this disseration, rather than imposing constraints on the controller or corresponding Youla parameter, we instead consider directly designing the *closed-loop mappings*. It is well-known that the set of all achievable closed-loops for a given plant is affine linear [15], and additional convex structural constraints preserve convexity of this subspace. This is the theme followed by the recently developed System Level Synthesis (SLS) framework [11], which employs an affine subspace constraint to *implicitly* parameterize the set of all achievable stabilized closed-loop mappings for a given plant [11]. Moreover, [11] has demonstrated that the corresponding controller has an implementation that inherits the structure imposed on the closed-loops.

A potential limitation of the SLS framework is that the *implicit* affine constraint utilized is infinite dimensional and is typically enforced numerically, requiring temporal FIR approximations. The first contribution of this dissertation is the derivation of an *explicit* parameterization of the set of all achievable stabilized closed-loop mappings. This result is derived in the state feedback setting, in the case that (a) the open-loop plant dynamics are stable, or (b) the mapping from control to state in open-loop is invertible, or (c) open-loop dynamics are decoupled. By providing an explicit (rather than implicit) parameterization, we eliminate the need for FIR numerical approximations and allow for the derivation of analytic IIR solutions.

We utilize this explicit parameterization to analyze (possibly infinite-extent) spatially-invariant problems. Our motivation for analyzing this class of systems is that it allows us to provide insight regarding fundamental limitations of controller performance with system size and/or degree of locality imposed. Although most real-life systems have finite spatial extent, we note that the infinite-spatial-extent spatially-invariant setting often provides useful approximations for large but finite systems [16] and provides useful insights in many cases. This is a widely accepted argument when considering time dependence; infinite time horizon problems are a useful idealization for control problems over a long, but finite time horizon. In addition, the infinite-extent problem is certainly useful when it yields computational and analytical insight, as is shown in this paper. The case of vehicular platoons [3] provides one compelling example of how the infinite-extent control problem gives sharp insight into issues that arise in the large-but-finite setting.

This structured closed-loop design problem provides one method for designing a controller with a *structured realization*. In this dissertation, we highlight that the optimal controller design problem subject to structural constraints on the closed-loop transfer matrices provides a *convex relaxation* of the optimal controller design problem subject to constraints on the structure of a state space realization (implementation). The second key contribution of this dissertation is a step toward quantifying the corresponding performance gap by demonstrating that the the optimal controller design problem subject to sparsity constraints on the closed-loop transfer matrices is *infeasible* when additional relative feedback constraints are imposed.

We next look at problem instances where the unconstrained optimal control policy is inherently localized. It was demonstrated in [16, 17] that for certain classes of infinite-extent systems formulated over an L_2 state space, the optimal feedback is inherently localized. We continue in this line of work by providing similar results for the case that the underlying state space is a Sobolev space. The third key contribution of this dissertation is a proof that the optimal LQR feedback for a PDE over a Sobolev space will be a static (in time) state feedback that decays exponentially in space.

We note that PDEs can be thought of as spatially distributed systems with an uncountable number of 'subsystem components'. Relevant applications in this setting include fluids [18, 19] and biological applications. The PDE setting provides a continuum approximation of systems over a discrete domain as well; the heat equation can be thought of as the continuum limit of a chain of first order subsystems, and the wave equation represents a continuum limit of consensus of a chain of second order subsystems (e.g. vehicular platoons and AC power networks). Motivated by the study of optimal LQR control of PDEs, the fourth key contribution of this dissertation is a derivation of an affine linear parameterization of all achievable stabilized closed-loop mappings for a PDE. We further demonstrate that this parameterization, along with parameterizations provided in earlier chapters, are special cases of a general operator based framework of SLS.

1.1 Contributions & Organization

The contributions of this dissertation, organized by chapter, are as follows:

Chapter 2: An Explicit Parameterization of Closed-Loops of Spatially-Distributed Systems

1. Motivated by the *implicit* parameterizations of all achievable stabilized closed-loops provided by SLS, we provide an *explicit* parameterization of all achievable stabilized closed-loops for finite-dimensional systems and spatially-invariant systems over a countable state space for both continuous and discrete-time, provided at least one of the following three assumptions hold:

   (a) The open-loop dynamics are stable,

   (b) The mapping from control to state in open-loop is invertible,

   (c) The open-loop dynamics are decoupled and controllable.

2. These affine linear parameterizations allow for conversion of the optimal $\mathcal{H}_2$ design problem with closed-loop structural constraints to be converted to a standard model matching problem, and are employed to provide:

   - Analytic solutions to a first-order consensus problem with closed-loop sparsity constraints,
   - Numerical solutions to a vehicle platoon control problem.

These examples are shown to provide information about how performance limitations scale with system parameters and number of subsystems.

Chapter 3: Controller Structure vs. Closed-Loop Structure of Spatially-Distributed Systems

1. We demonstrate that the optimal controller design problem subject to sparsity constraints on the closed-loop transfer functions provides a *convex relaxation* of the optimal controller design problem subject to a constraint that the controller has a structured state-space realization.

2. We provide a compact, convex characterization of a *relative feedback* controller design constraint. We demonstrate that the relative feedback requirement can be written in terms of the closed-loop mappings in certain problem settings.

3. We show that when relative feedback constraints are imposed, the optimal controller design problem subject to closed-loop transfer function sparsity constraints may be *infeasible*.

Chapter 4: Control of Spatially-Distributed Systems over Sobolev Spaces

1. We demonstrate that the LQR design problem for a spatially-invariant system with state space as a Sobolev space can be converted to an equivalent problem over an L_2 state space.

2. We demonstrate that the optimal state-feedback for the LQR design problem in this Sobolev space setting is a spatial convolution operator whose kernel decays exponentially. This allows for analysis of PDEs with higher order dynamics which could not be analyzed by previous techniques that considered only an L_2 setting.

3. Results are applied to analyze the LQR design problem for the wave equation over the real line with a cost functional representing mechanical energy, which is related to (but not equal to) the underlying Sobolev space norm.

Chapter 5: An Operator Perspective of System Level Synthesis

1. We provide an operator framework for the SLS controller design methodology, which allows for analysis of continuous and discrete-time systems over a general Banach space.

2. We highlight that the parameterizations provided in Chapter 2 are special cases of this more general framework.

3. These general operator based parameterizations are specialized to the setting of control of PDEs. As a case study we employ these parameterizations to solve the unconstrained LQR design problem for the diffusion equation over the real line.

Chapter 2

An Explicit Parameterization of Closed-Loops of Spatially-Distributed Systems

Abstract **- In this chapter, we study controller design problems for spatially distributed systems in which spatial sparsity constraints on the closed-loop mappings are imposed. This is done in the context of the System Level Synthesis (SLS) framework, which employs affine subspace constraints to implicitly parameterize the set of all stabilized closed loops. While in recent SLS-based designs those constraints are handled numerically using FIR closed-loop representations, we exhibit an *explicit* parameterization that allows for more efficient IIR representations of the temporal part of the closed-loop dynamics. We consider two distinct classes of distributed systems:**

1. **Systems (possibly with coupled subsystem dynamics) with either**

   (a) **stable open-loop dynamics, or**

   (b) **invertible control to state mapping (i.e. invertible "B" state space matrix).**

   (This class of systems is analyzed in 'Part 1' of this chapter).

2. **Systems with *decoupled* (open-loop) subsystem dynamics that are controllable, allowing for unstable dynamics and non-invertible control to state operator. (This class of systems is analyzed in 'Part 2' of this chapter).**

For this second class of systems, we begin by considering first order dynamics and then employ a backstepping-like procedure to extend to the higher order setting. For the class of spatially-invariant systems (of either type (1) or (2)), we show that the SLS design problem can be reduced to a finite-dimensional

model matching problem even in the infinite spatial extent setting. For case (1), we provide numerical results for the control of an infinite chain of first-order subsystems where each subsystem's dynamics are coupled with that of their nearest neighbors' in open-loop. For case (2), we provide examples from consensus algorithms and vehicular formations for which this formulation leads to analytical solutions of the spatially sparse $\mathcal{H}^2$ design problem.

This Chapter is based on the following Publications:
[20] - E. Jensen and B. Bamieh, Optimal spatially-invariant controllers with locality constraints: A system level approach, in 2018 Annual American Control Conference (ACC), pp. 2053–2058, IEEE, 2018.
[21] - E. Jensen and B. Bamieh, A backstepping approach to system level synthesis for spatially invariant systems, in 2020 Annual American Control Conference (ACC), pp. 5295–5300, IEEE, 2020.
[22] - E. Jensen and B. Bamieh, An explicit parameterization of closed loops for spatially-invariant controllers with spatial sparsity constraints, IEEE Transactions on Automatic Control (To Appear).

2.1 Introduction

We follow the System Level Synthesis (SLS) method [11] by directly designing the closed-loop mappings rather than the controller or corresponding Youla parameter. We note that SLS employs an affine subspace constraint to *implicitly* parameterize the set of all achievable stabilized closed-loop mappings for a given plant. Structural constraints can be imposed on these closed-loop mappings in a convex manner, and the corresponding controller has an implementation that inherits this closed-loop structure. However, this affine subspace constraint is infinite dimensional, and is typically enforced numerically, requiring temporal FIR approximations.

In this chapter, we derive *explicit* parameterizations of the set of all achievable stabilized closed-loop mappings for certain subclasses of spatially distributed systems. By providing an explicit (rather than implicit) parameterization, we eliminate the need for FIR numerical approximations and allow for the derivation of analytic IIR solutions.

In the setting of spatially-invariant systems (with finite or infinite spatial dimension), we demonstrate that our parameterization allows the optimal $\mathcal{H}_2$ design problem to be written as a standard unconstrained model matching problem. When finite spatial sparsity constraints are imposed on the closed-loops, this unconstrained model matching problem has *finitely many* transfer function parameters, even in the infinite-spatial-extent setting.

Although most real-life systems have finite spatial extent, we note that the infinite-spatial-extent spatially-invariant setting often provides useful approximations for large but finite systems [16] and provides useful insights in many cases, e.g. for the vehicular platoons problem as demonstrated in [3].

We apply our parameterizations to derive analytic expressions for the optimal $\mathcal{H}_2$ performance and corresponding controller implementation for the consensus problem when spatial sparsity constraints on the closed-loops are imposed, and we provide numerical results for the vehicular platoon problem. Our approach allows us to provide commentary about the performance limitations and scalings with system size for these applications, complementing analysis of recent works including [23–25].

This chapter is structured as follows. We begin by introducing some preliminaries on spatially-invariant systems of finite or infinite spatial extent in Section 2.2. In Section 2.3, we state the structured $\mathcal{H}_2$ design problem of interest, present a closed-loop design procedure in this setting, and introduce motivating applications.

In Part 1 of this chapter (Section 2.4), we analyze distributed systems (with possibly coupled subsystem dynamics, in which the open loop is stable or the mapping from control to state is invertible. We derive an *explicit* parameterization of all stabilized closed loops for this class of systems, and demonstrate that this parameterization allows the $\mathcal{H}_2$ design problem with closed-loop constraints to be converted to a standard model matching problem with finitely many transfer function parameters.

In Part 2 of this Chapter (Section 2.5), we derive an *explicit* parameterization of all stabilized closed loops for spatially-invariant systems with uncoupled first-order subsystem dynamics in Section 2.5.2 and extend this to the case of higher-order subsystems in Section 2.5.3. We demonstrate that this parameterization can be used to convert the structured $\mathcal{H}_2$ design problem to a standard model matching problem. We apply our results to the consensus problem in Sections 2.5.4-2.5.5 and to the vehicle formation problem in Section 2.5.6. We extend our results to the case of spatially-varying systems in Section 2.5.7.

2.2 Notation & Preliminaries

We consider both continuous and discrete-time settings. We let $\mathcal{R}$ denote the set of proper and rational (possibly matrix-valued) transfer functions, and let $\overline{\mathcal{R}} \subset \mathcal{R}$ denote the subspace of strictly proper transfer functions. We define a transfer function to be stable if

- **continuous-time** it has all poles in the open left half plane $\{s : \Re(s) < 0\}$,
- **discrete-time** it has all poles inside the open unit disk $\{z : |z| < 1\}$.

We let $\mathcal{R}_s$ and $\overline{\mathcal{R}}_s$ denote the subsets of stable elements of $\mathcal{R}$ and $\overline{\mathcal{R}}$ respectively. The $\mathcal{H}_2$ norm provides one measure of a system in $\overline{\mathcal{R}}_s$, and is defined as follows:

- **continuous-time systems:**

$$\|H\|_{\mathcal{H}_2}^2 := \text{tr}\left(\int_{-\infty}^{\infty} G(i\omega)^* G(i\omega) d\omega\right)$$

- **discrete-time systems:**

$$\|H\|_{\mathcal{H}_2}^2 := \frac{1}{2\pi}\text{tr}\left(\int_0^{2\pi} H(e^{i\theta})^* H(e^{i\theta}) d\theta\right)$$

We consider spatially distributed dynamical (i.e. spatio-temporal) systems where the state and all external signals are functions of time t and a (discrete) spatial variable n. We denote such spatio-temporal signals using lower case letters; for example, the (possibly vector-valued) state at location n and time t is denoted by either

$$x(n,t) \;\text{ or }\; x_n(t), \qquad n \in \mathbb{G},\; t \in \mathcal{T}$$

where the spatial index n takes values in the finite set $\mathbb{G} = \mathbb{Z}_N$ or the countably infinite set $\mathbb{G} = \mathbb{Z}$. We allow for discrete time systems ($\mathcal{T} = Z^+ := \{0, 1, ...\}$) as well as continuous time systems ($\mathcal{T} = \mathbb{R}^+ := [0, \infty)$). With some abuse of notation, we also denote the (temporal) Laplace or Z-transform (Transfer Function representation) of a spatio-temporal signal by a lower case letter

$$\begin{aligned} x_n(s) \;\text{ or }\; x(n,s) &:= \int_0^\infty x_n(t)\; e^{-st} dt, \\ x_n(z) \;\text{ or }\; x(n,z) &:= \sum_{t=0}^{\infty} x_n(t)\; z^{-t} dt, \end{aligned} \tag{2.1}$$

We use a (^) to denote the (spatial) Fourier transform of a spatio-temporal signal, i.e.

$$\hat{x}_\lambda(t) = \hat{x}(\lambda, t) := \sum_{n \in \mathbb{G}} x_n(t) e^{-in\lambda}, \quad \lambda \in \hat{\mathbb{G}}. \tag{2.2}$$

Such transforms of a matrix-valued signals are defined component-wise.

We also represent signals as finite or infinite vectors

$$\begin{aligned} x(t) &= \begin{bmatrix} x_0(t) \\ \vdots \\ x_{N\text{-}1}(t) \end{bmatrix}, \\ x(t) &= \begin{bmatrix} \cdots & x_{\text{-}1}^T(t) & x_0^T(t) & x_1^T(t) & \cdots \end{bmatrix}^T, \end{aligned}$$

depending on whether $\mathbb{G}$ is finite or infinite respectively, and similarly define the Laplace (or Z-) transforms of such signals $x(s)$ (or $x(z)$).

We use the $L^2(\mathbb{G} \times \mathcal{T})$ (or denoted simply as L^2) norm on (vector-valued) spatio-temporal signals, given by

$$\|x\|_2^2 := \sum_{n \in \mathbb{G}} \int_{t \in \mathcal{T}} x_n^*(t) x_n(t) dt,$$

where h is the (spatially-varying) impulse response sequence defining the system H, and this integral is respect to standard Lesbegue measure when $\mathcal{T} = \mathbb{R}^+$ and counting measure when $\mathcal{T} = \mathbb{Z}^+$.

A spatio-temporal system H is an operator mapping between spatio-temporal signals w and x,

$$x \;=\; H\,w,$$

which in the linear time-invariant case can be written as:

$$x(n,t) = \sum_{l\in\mathbb{G}} \int_{\tau\in\mathcal{T}} h(n,l,t-\tau)\; w(l,\tau)d\tau. \tag{2.3}$$

When the system is also *spatially invariant* [16], this relationship takes the form of a spatio-temporal convolution

$$x(n,t) = \sum_{l\in\mathbb{G}} \int_{\tau\in\mathcal{T}} h(n-l,t-\tau)w(l,\tau)d\tau \tag{2.4}$$

where $n-l$ is taken modulo N in the case that $\mathbb{G} = \mathbb{Z}_N$. Such *spatially-invariant* systems are readily defined in the transfer function domain as follows.

For simplicity of exposition, throughout the rest of this section we often present results in only continuous-time, and the discrete-time setting holds as well unless otherwise stated.

Definition 2.2.1 *A* spatially-invariant system *is an operator H on $L^2(\mathbb{G}\times\mathbb{R}^+)$ which can be represented by spatial convolution in the transfer function domain for each fixed frequency s, i.e.*

$$\begin{aligned} x_n(s) = (HU)_n(s) &:= (h(s) * u(s))_n \\ &:= \sum_{m\in\mathbb{G}} h_m(s)u_{n-m}(s), \end{aligned} \tag{2.5}$$

with each $h_m(s)$ a finite-dimensional transfer matrix. Note that we use $(*)$ to denote both circular convolution when $\mathbb{G} = \mathbb{Z}_N$ and standard discrete convolution when $\mathbb{G} = \mathbb{Z}$, taking $n-m$ to be computed with the group operation on $\mathbb{G}$, e.g. standard subtraction in $\mathbb{Z}$ and subtraction mod N in $\mathbb{Z}_N$.

The spatially-invariant system $H(\cdot)$ is completely specified by the (possibly infinite) sequence of transfer functions $\{h_m(\cdot)\}_{m\in\mathbb{G}}$, which we refer to as the convolution kernel of $H(\cdot)$. We say a spatially-invariant system $H = \{h_m(\cdot)\}_{m\in\mathbb{G}} \in \mathcal{R}$ (resp. $\mathcal{R}_s$, $\overline{\mathcal{R}}$, $\overline{\mathcal{R}}_s$) if each element of the convolution kernel $h_m \in \mathcal{R}$ (resp. $\mathcal{R}_s$, $\overline{\mathcal{R}}$, $\overline{\mathcal{R}}_s$).

In the finite space setting $\mathbb{G} = \mathbb{Z}_N$, a spatially-invariant system H can be represented as a circulant matrix, e.g.

$$x(s) = \begin{bmatrix} h_0(s) & h_{N-1}(s) & \cdots & h_1(s) \\ h_1(s) & h_0(s) & \cdots & h_2(s) \\ \vdots & & \ddots & \\ h_{N-1}(s) & h_{N-2}(s) & \cdots & h_0(s) \end{bmatrix} u(s), \tag{2.6}$$

and in the infinite space setting $\mathbb{G} = \mathbb{Z}$, can be represented by an infinite-dimensional Toeplitz matrix, e.g.

$$x(s) = \begin{bmatrix} \ddots & \ddots & \ddots & & \\ & h_1(s) & h_0(s) & h_{-1}(s) & \\ & & h_1(s) & h_0(s) & h_{-1}(s) \\ & & \ddots & \ddots & \ddots \end{bmatrix} u(s). \tag{2.7}$$

By taking a Fourier transform in the spatial domain, the spatially invariant system (2.5) can be written as

$$\hat{x}_\lambda(s) \;=\; \hat{h}_\lambda(s)\; \hat{w}_\lambda(s), \;\; \lambda \in \hat{\mathbb{G}}. \tag{2.8}$$

Example 2.2.1 *The* temporal differentiation operator*, defined in the time domain by*

$$\dot{x}_n(t) := \frac{d}{dt} x_n(t),$$

is a spatially-invariant system. It can be represented in the transfer function domain as multiplication by the diagonal (potentially infinite dimensional) matrix sI:

$$\dot{x}_n(t) \;\leftrightarrow sI \cdot x_n(s).$$

Similarly, the temporal shift *operator:*

$$x_n(t) = x_n(t+1),$$

is a spatially-invariant system and can be represented in the transfer function domain as multiplication by the diagonal (potentially infinite dimensional) matrix zI.

Special classes of spatially-invariant systems include:

- *Pointwise multiplication operators*:

  $B = \{b_n\}_{n\in\mathbb{G}}$ is a *pointwise multiplication operator* if

  $$b_n = \begin{cases} 0, \text{for all } n \neq 0, \\ b_0 \text{ (a static matrix) for } n = 0. \end{cases}$$

  We use B to denote the operator and b to denote the static matrix $b = b_0$ which defines B. Pointwise multiplication operators are represented by static and block diagonal circulant (or Toeplitz) matrices.

- *Spatial convolution operators*: C is a *spatial convolution operator* if it is of the form

  $$(Cx)_n(t) := (c * x)_n(t) = \sum_{m\in\mathbb{G}} c_m x_{n-m}(t),$$

  with $\{c_m\}$ a sequence of real-valued matrices, i.e. the convolution kernel of C is composed of static matrices. Spatial convolution operators are represented by *static* circulant (or Toeplitz) matrices.

2.2.1 Locality Constraints

We are interested in the design of *localized* systems, i.e. systems for which the output at each spatial location is computed using only information from nearby spatial locations. The following definitions formalize the specific class of locality constraints we focus on throughout this chapter.

Definition 2.2.2 *Let H be a real-valued matrix or transfer matrix, block partitioned as*

$$H = \begin{bmatrix} H_{11} & H_{12} & \cdots & H_{1n} \\ H_{21} & H_{22} & \cdots & H_{2n} \\ \vdots & & & \\ H_{n1} & H_{n2} & \cdots & H_{nn} \end{bmatrix}.$$

We say that H has band size M *if the (i, j) block, H_{ij}, is zero whenever $|i-j| > M$. Similarly, a spatially-invariant system H (of finite or infinite spatial extent) has* band size M *if the mapping from input u to output x can be written as*

$$x_n(s) = \sum_{|m|\leq M} H_m(s) u_{n-m}(s),$$

i.e. the convolution kernel defining H has entries $H_n(s) \equiv 0$ for $|n| > M$.

When H is defined on vector-valued signal spaces we define the band size of H component-wise, e.g. if $u_n(s) \in \mathbb{R}^P$, $x_n(s) \in \mathbb{R}^L$, and H has band size M, then for each i the mapping to the i^{th} output component x^i is of the form:

$$x_n^i(s) = \sum_{|m|\leq M} \sum_{j=1}^{P} H_m^{ij}(s) u_{n-m}^j(s).$$

Note that the definition of band size of a system depends only on its *transfer function* representation, and is independent of the chosen *state-space realization.* In addition, the choice of the specific banded structure can be viewed as corresponding to an underlying communication graph structure of a line graph with links between m nearest neighbors. In Chapter 3, we consider other types of structure, and generalize notions of structure to other communication graph structures.

Example 2.2.2 *By definition, all pointwise multipliation operators have band size 0. The temporal differentiation operator and the temporal shift operator also have band size 0.*

We summarize some useful properties of spatially-invariant systems in the following proposition.

Proposition 2.2.1 *Let K and H be spatially-invariant systems (of finite or infinite spatial extent). Assume K, H are defined on signal spaces of appropriate dimensions so that the composition operator $(KH)(x) = K(H(x))$ is well-defined. Then the following hold:*

1. *KH is a spatially-invariant system. In particular, for each positive integer n, K^n denotes the composition of K n times and is a spatially-invariant system.*
2. *If K and H have finite band sizes M and N, respectively, then KH has band size $N + M$. Thus, the composition of two spatially-invariant systems with finite band size also has finite band size.*
3. *The inverse operator K^{-1}, when it exists, is also a spatially-invariant system.*

The structural properties of circulant and Toeplitz transfer matrices simplify the computation of the $\mathcal{H}_2$ norm in the spatially-invariant setting.

2.2.2 $\mathcal{H}_2$ Norm of Spatially-Invariant Systems

Given a spatially-invariant system $H = \{h_m(s)\}_{m\in\mathbb{G}} \in \overline{\mathcal{R}}_s$, the $\mathcal{H}_2$ norm of the n^{th} row of the representation (2.6) or (2.7) of H corresponds to the L^2 norm of the output at spatial site n subject to impulse disturbances at all inputs. Similarly, the $\mathcal{H}_2$ norm of column n corresponds to the L^2 norm of all outputs subject to an impulse disturbance at spatial site n. The $\mathcal{H}_2$ norm of any row or any column will be equivalent and can be computed as:

- Finite Space Setting ($\mathbb{G} = \mathbb{Z}_N$):
$$\sum_{m=0}^{N-1} \|h_m\|_{\mathcal{H}_2}^2 = \|He_j\|_{\mathcal{H}_2}^2, \tag{2.9}$$
- Infinite Space Setting ($\mathbb{G} = \mathbb{Z}$), with finite band size M:
$$\sum_{m=-M}^{M} \|h_m\|_{\mathcal{H}_2}^2 = \sum_{m\in\mathbb{Z}} \|h_m\|_{\mathcal{H}_2}^2 = \|He_j\|_{\mathcal{H}_2}^2, \tag{2.10}$$

where He_j denotes the product of the circulant matrix H with the j^{th} standard basis vector in $\mathbb{R}^N$ in the finite space setting (e.g. $e_1 := \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$). In the infinite space setting, He_j denotes the product of the infinite-dimensional Toeplitz matrix H with the infinite array $e_j = \begin{bmatrix} \cdots e_j(j-1) & e_j(j) & e_j(j+1) & \cdots \end{bmatrix}^T = \begin{bmatrix} \cdots 0 & 1 & 0 & \cdots \end{bmatrix}^T$, or equivalently He_j is the convolution of $\{h_n\}_{n\in\mathbb{Z}}$ with the sequence defined by the infinite vector e_j. Note that the choice of j is arbitrary due to spatial invariance. In the finite space setting ($\mathbb{G} = \mathbb{Z}_N$),
$$\|H\|_{\mathcal{H}_2}^2 = N \cdot \|He_j\|_{\mathcal{H}_2}^2.$$

We refer to (2.9) and (2.10) as the $\mathcal{H}_2$ norm *per spatial site* of a spatially-invariant system.

2.3 Problem Formulation & Motivating Applications

We consider spatially distributed systems whose dynamics can be represented in the form:

$$\begin{aligned} x(t+1) \;/\; \dot{x}(t) &= (Ax)(t) + (B_1 w)(t) + (B_2 u)(t), \\ \overline{z}(t) &= (C_1 x)(t) + (D_{12} u)(t). \end{aligned} \tag{2.11}$$

where x, u, w and $\overline{z}$ are all spatio-temporal signals which represent the spatially distributed state, control action, external disturbance, and performance output of the system respectively, e.g. $x_n(t)$ represents the state of the system at spatial index $n \in \mathbb{G}$ at time $t \in \mathcal{T} = \mathbb{R}^+$ or $\mathbb{Z}^+$. We analyze both

- *Finite space settings:* $\mathbb{G} = \mathbb{Z}_N$ (the undirected torus) representing N systems on a ring, and
- *Infinite space settings:* $\mathbb{G} = \mathbb{Z}$ representing a chain of subsystems,

considering the infinite space setting $\mathbb{G} = \mathbb{Z}$ to provide insight into the large but finite space setting, $\mathbb{G} = \mathbb{Z}_N$, as the number of subsystems $N \to \infty$. For simplicity, we restrict to an odd number of subsystems N in the finite space setting, so that band size constraints will be symmetric. When $\mathbb{G} = \mathbb{Z}_N$, A, B_1, B_2, C_1, and D_{12} can all be represented by finite-dimensional matrices; in the spatially-invariant setting, these matrices are circulant. When $\mathbb{G} = \mathbb{Z}$, we assume that A, B_1, B_2, C_1, and D_{12} are spatially-invariant systems.

Given a (dynamic or static) state-feedback controller $u = Kx$ for system (2.11), we formally define the corresponding closed-loop mappings Φ^x and Φ^u from disturbance $B_1 w$ to state x and control action u as follows:

Finite Space Systems

Let A, B_1, B_2 be finite-dimensional scalar-valued matrices. Then, following [11], the closed-loop transfer functions Φ^x, Φ^u are given by:

$$\begin{aligned} \begin{bmatrix} x(s) \\ u(s) \end{bmatrix} &= \begin{bmatrix} \Phi^x(s) \\ \Phi^u(s) \end{bmatrix} B_1 w(s) \\ &:= \begin{bmatrix} (sI - A - B_2 K(s))^{-1} \\ K(s)(sI - A - B_2 K(s))^{-1} \end{bmatrix} B_1 w(s). \end{aligned} \tag{2.12}$$

Spatially-Invariant Systems (Finite or Infinite Space Setting)

Let A, B_1, B_2 be spatial convolution operators, and assume that K is a spatially-invariant system. Then, following [20], the corresponding closed-loop mappings Φ^x, Φ^u are spatially-invariant systems given by:

$$\begin{aligned} \begin{bmatrix} x \\ u \end{bmatrix} &= \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 w \\ &:= \begin{bmatrix} (sI - A - B_2 K)^{-1} \\ K(sI - A - B_2 K)^{-1} \end{bmatrix} B_1 w \end{aligned} \tag{2.13}$$

The controller which results in the spatially-invariant closed-loop mappings defined in (2.12) or (2.13) is given by

$$K = \Phi^u \left(\Phi^x\right)^{-1},$$

and we say that K is *stabilizing* for (2.11) if the resulting closed-loop mappings Φ^x, Φ^u, defined in (2.12) or (2.13), are elements of $\overline{\mathcal{R}}_s$ [11], [20].

Proposition 2.3.1 *Let $u = Kx$ be a controller for* (2.11) *with A, B_1, and B_2 spatial convolution operators. Then K is a spatially-invariant system if and only if Φ^x, Φ^u defined in* (2.13) *are spatially-invariant.*

Proof: This follows from a direct application of Proposition 2.2.1. ■

2.3.1 $\mathcal{H}_2$ Performance Metrics

We model $\{w_n\}$ as a mutually uncorrelated white stochastic process. The steady-state variance of the fluctuation of the performance output $\overline{z}$ provides a measure of performance that can be calculated as the $\mathcal{H}_2$ norm of the closed-loop system [26]:

$$\|\mathcal{F}(P;K)\|_{\mathcal{H}_2}^2 = \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 \right\|_{\mathcal{H}_2}^2 . \tag{2.14}$$

In the spatially-invariant setting, optimizing the $\mathcal{H}_2$ norm of the system (2.14) is equivalent to optimizing the $\mathcal{H}_2$ norm *per spatial site* (see equations (2.9) - (2.10)). The $\mathcal{H}_2$ design problem of interest is stated in terms of the $\mathcal{H}_2$ norm per site as follows:

Optimal $\mathcal{H}_2$ controller design for spatially-invariant systems with closed-loop spatial sparsity constraints:

$$\left.\begin{array}{ll} \inf\limits_{K \text{ stabilizing}} & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 e_j \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & K \text{ spatially-invariant}, \\ & \text{System dynamics (2.11)}, \\ & \Phi^u, \Phi^x \text{ have band size } M. \end{array}\right\} \tag{2.15}$$

The constrained design problem (2.15) is *convex*. Without the band size constraints, this problem is a standard $\mathcal{H}_2$ design problem, which can be solved analytically via the algebraic Riccati equation in the finite space setting and using the techniques of [16] in the infinite space setting. Imposing this band size constraint makes the problem (2.15) different and has the following consequences:

1. If Φ^u and Φ^x have band size M, then the corresponding controller has an implementation that inherits this structure, restricting subcontroller communication to a neighborhood of size M,

2. The constrained problem (2.15) can be converted to a standard *finite-dimensional* model-matching problem with $(2M + 1)$ transfer function parameters, with M the constrained closed-loop band size. This holds even in the infinite space setting, provided C_1 and D_{12} have finite band size.

2.3.2 System Level Parameterizations

[11] provided a simple affine parameterization of all achievable stabilized closed-loop maps, for a *finite-dimensional* system of the form (2.11). The continuous-time analogue holds as well, as shown in Chapter 5.

Lemma 2.3.2 *A (static or dynamic) state feedback controller* $u = Kx$ *is stabilizing for the finite-dimensional discrete-time system* (2.11) *if and only if the corresponding closed-loop mappings* Φ^x *and* Φ^u *are elements of the affine subspace defined by:*

$$\left.\begin{array}{l} \begin{bmatrix} (zI - A) & -B_2 \end{bmatrix} \begin{bmatrix} \Phi^x(z) \\ \Phi^u(z) \end{bmatrix} = I, \\ \Phi^x \ , \ \Phi^u \in \overline{\mathcal{R}}_s \end{array}\right\} \tag{2.16}$$

The following Lemma provides a generalization of the finite dimensional SLS closed-loop parameterization (2.16) to the (finite or infinite extent) spatially-invariant setting. For simplicity of exposition, this result is stated only in discrete-time, but the continuous-time analogue holds as well.

Lemma 2.3.3 *A spatially invariant state-feedback control policy* $u = Kx$ *is stabilizing for the spatially-invariant system* (2.11) *if and only if the corresponding closed-loop mappings* Φ^x *and* Φ^u *are spatially-invariant systems which are elements of the affine subspace defined by:*

$$\left.\begin{array}{l} \begin{bmatrix} (zI - A) & -B_2 \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} = I, \\ \Phi^x \ , \ \Phi^u \in \overline{\mathcal{R}}_s \end{array}\right\} \tag{2.17}$$

Proof: A spatial Fourier transform (2.2) allows (2.23) to be written equivalently as a *decoupled* family of finite-dimensional systems, indexed by $\lambda \in \hat{\mathbb{G}}$.

$$\hat{x}_\lambda(t+1) \ / \ \frac{d}{dt}\hat{x}_\lambda(t) = \hat{a}_\lambda \hat{x}_\lambda(t) + (\hat{b}_1)_\lambda \hat{w}_\lambda(t) + (\hat{b}_2)_\lambda \hat{u}_\lambda(t), \tag{2.18}$$

where $\hat{a}_\lambda = \sum_{n \in \mathbb{G}} a_n e^{-in\lambda}$ is the transform of the convolution kernel defining A, and similarly for $(\hat{b}_1)_\lambda$ and $(\hat{b}_2)_\lambda$. By the finite dimensional SLS results [11], the state feedback control

$$\hat{u}_\lambda(z) = \hat{k}_\lambda(z)\hat{x}_\lambda(z)$$

is internally stabilizing for (2.18) if and only if the resulting transfer functions $\hat{\phi}^x_\lambda$ and $\hat{\phi}^u_\lambda$ defined by

$$\begin{aligned} \begin{bmatrix} \hat{x}_\lambda(z) \\ \hat{u}_\lambda(z) \end{bmatrix} &= \begin{bmatrix} \hat{\phi}^x_\lambda(z) \\ \hat{\phi}^x_\lambda(z) \end{bmatrix} (\hat{b}_1)_\lambda \hat{w}_\lambda(z) \\ &= \begin{bmatrix} (zI - \hat{a}_\lambda - (\hat{b}_2)_\lambda \hat{k}_\lambda(z))^{-1} \\ \hat{k}_\lambda(z)(zI - \hat{a}_\lambda - (\hat{b}_2)_\lambda \hat{k}_\lambda(z))^{-1} \end{bmatrix} (\hat{b}_1)_\lambda \hat{w}_\lambda(z) \end{aligned} \tag{2.19}$$

are in the affine subspace

$$\left.\begin{array}{l} \begin{bmatrix} zI - \hat{a}_\lambda & -(\hat{b}_2)_\lambda \end{bmatrix} \begin{bmatrix} \hat{\phi}^x_\lambda(z) \\ \hat{\phi}^x_\lambda(z) \end{bmatrix} = I, \\ \hat{\phi}^x_\lambda, \ \hat{\phi}^u_\lambda \in \overline{\mathcal{R}}_s. \end{array}\right\} \tag{2.20}$$

By the results of [16], the state feedback

$$u = Kx,$$

with K a spatio-temporal system with convolution kernel $\{k_n(z)\}$ internally stabilizes the distributed system (2.23) if and only if $\hat{k}_\lambda$ is internally stabilizing for (2.18) for each $\lambda \in \hat{\mathbb{G}}$. Thus, the explicit parameterizations provided for finite dimensional systems may be applied to the finite dimensional system (2.18) for each $\lambda \in \hat{\mathbb{G}}$. ■

In addition, the corresponding controller $u = Kx$ can be implemented as follows, inheriting the structure of the closed-loops Φ^x and Φ^u.

$$\begin{aligned} u &= s\Phi^u(x - \tilde{x}) \\ \tilde{x} &= (s\Phi^x - I)(x - \tilde{x}). \end{aligned} \tag{2.21}$$

A direct consequence of Lemma 2.3.3 is that problem (2.15) can be rewritten in terms of closed-loop mappings as:

$$\left.\begin{array}{rll} J_M^{\text{opt}} = & \inf\limits_{\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s} & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 e_j \right\|^2_{\mathcal{H}_2} \\ & \text{s.t.} & \begin{bmatrix} (zI - A) & -B_2 \end{bmatrix} \begin{bmatrix} \Phi^x(z) \\ \Phi^u(z) \end{bmatrix} = I, \\ & & \Phi^x, \Phi^u \text{ spatially-invariant}, \\ & & \Phi^u, \Phi^x \text{ have band size } M. \end{array}\right\} \tag{2.22}$$

Note that (2.22) contains an *implicit* affine constraint on the closed-loops, which has been handled in the finite-dimensional setting in e.g. [11] numerically with temporal FIR constraints. In order to compute analytic solutions and allow for IIR representations, we instead develop an equivalent *explicit parameterization*.

2.4 Part 1: Coupled Subsystem Dynamics

We look at a subclass of systems with state dynamics of the form

$$x(t+1) \ / \ \dot{x}(t) = (Ax)(t) + (B_1 w)(t) + (B_2 u)(t), \tag{2.23}$$

where either

1. (2.23) is finite-dimensional, with A, B_1, B_2 all finite-dimensional constant matrices,

2. (2.23) is spatially-invariant, over a finite ($\mathbb{G} = \mathbb{Z}_N$) or countably infinite ($\mathbb{G} = \mathbb{Z}$) spatial domain. In this case, we assume that A is spatial convolution operator whose convolution kernel $\{a_n\}$ has only finitely many non-zero entries, and B_1, B_2 are operators of pointwise multiplication by the static matrices b_1, b_2.

with A a spatial convolution operator (with only finitely many non-zero elements in its convolution kernel $\{a_n\}$) and B_1, B_2 pointwise multiplication operators.

We consider two distinct cases in Part 1 of this chapter:

1. The open loop system:
$$x(t+1)/\dot{x}(t) = (Ax)(t),$$
is stable, or

2. The mapping B_2 from control to state is an *invertible* operator.

(2.17) provides an *implicit* parameterization of all achievable, stabilized closed-loops for system (2.23) in any of these settings. In this section, we derive instead an *explicit* parameterization, leveraging the fact that the affine subspace constraint (2.17) can be rearranged to write one of the closed-loops in terms of the other:

$$\begin{aligned} \Phi^u &= -B_2^{-1}\left((sI - A)\Phi^x - I\right) && (B_2 \text{ invertible}), \\ \Phi^x &= (sI - A)^{-1}\left(I + B_2\Phi^u\right) && (A \text{ stable}). \end{aligned}$$

Finite-Dimensional Systems: Stable Open-loop Dynamics

We first consider the case that the open-loop system is stable, in the sense that all eigenvalues of A are in the open left half plane (for continuous-time) or in the open unit disk (for discrete-time). We present the following result in the discrete-time setting, noting that the continuous-time setting for this case is of the same form (just replace z with s in the following formulas). Using (2.34), we write Φ^x in terms of Φ^u as

$$\Phi^x(z) = (zI - A)^{-1}\left(I + B_2\Phi^u(z)\right). \tag{2.24}$$

From this formula, it follows that stability and strict-properness of Φ^u imply stability and strict properness of Φ^x. This leads to an explicit parameterization of all stabilized closed-loops, presented in the following theorem.

Theorem 2.4.1 *Consider the finite-dimensional, continuous or discrete-time system* (2.23)*, and assume that the matrix A has all eigenvalues in the stability region. Then $K(z)$ is a stabilizing state-feedback for* (2.23) *if and only if $\Phi^u \in \overline{\mathcal{R}}_s$ and*

$$\Phi^x(z) = (zI - A)^{-1}(I + B_2\Phi^u(z)). \tag{2.25}$$

Finite-Dimensional Systems: B_2 Invertible

We next consider the case that B_2 of (2.23) is a square, invertible matrix. In this case, we handle the discrete-time setting separately from the continuous-time setting. In the discrete-time setting, (2.34) allows Φ^u to be written in terms of Φ^x:

$$\begin{aligned} \Phi^u(z) &= B_2^{-1}\left((zI - A)\,\Phi^x(z) - I\right) \\ &= B_2^{-1}(z\Phi^x(z) - I) \;-\; B_2^{-1}A\Phi^x(z). \end{aligned} \tag{2.26}$$

From (2.26), we see that if Φ^x is stable, then Φ^u will be as well. To ensure strict properness of Φ^u, it is necessary that

$$\lim_{z\to\infty} z\Phi^x(z) = I, \tag{2.27}$$

to remove the DC component of (2.26). Thus, Φ^x must be of the form

$$\Phi^x(z) = \frac{1}{z}\left(I + \Theta(z)\right), \tag{2.28}$$

for some stable and strictly proper $\Theta(z)$. The preceding analysis proves the following result.

Theorem 2.4.2 *Consider the finite-dimensional, discrete-time system* (2.23) *and assume that B_2 is invertible. Then, $K(z)$ is a stabilizing state-feedback controller for* (2.23) *if and only if the closed-loop mappings Φ^x and Φ^u are of the form*

$$\begin{aligned} \Phi^x(z) &= \frac{1}{z}\left(I + \Theta(z)\right), \\ \Phi^u(z) &= \frac{1}{z}B_2^{-1}\big(zI - A\big)\Theta(z) \;-\; \frac{1}{z}B_2^{-1}A, \end{aligned} \tag{2.29}$$

for some $\Theta \in \overline{\mathcal{R}}_s$.

Theorem 2.4.2 provides an *explicit* parameterization of all closed-loops of (2.23) corresponding to a state-feedback controller which is spatially-invariant and stabilizing.

The continuous-time setting requires a bit more care, as multiplication by $\frac{1}{s}$ could introduce instability to the formulas (2.29), unlike multiplication by $\frac{1}{z}$. In this case, we employ the following Lemma, which follows from rewriting the standard affine subspace constraint (2.16).

Lemma 2.4.3 *The (dynamic or static) controller $u = Kx$ is stabilizing for* (2.61) *if and only if the resulting closed-loop mappings $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ satisfy the following affine constraint:*

$$\begin{bmatrix} (sI + I) - (A + I) & -B_2 \end{bmatrix} \begin{bmatrix} \Phi^x(s) \\ \Phi^u(s) \end{bmatrix} = I. \tag{2.30}$$

Then, following a similar procedure as the discrete-time setting leads to the following explicit parameterization of all stabilized closed-loops in the continuous-time setting.

Theorem 2.4.4 *Consider the finite-dimensional, continuous-time system* (2.23) *and assume that B_2 is invertible. Then, $K(s)$ is a stabilizing state-feedback controller for* (2.23) *if and only if the closed-loop mappings Φ^x and Φ^u are of the form*

$$\begin{aligned} \Phi^x(s) &= \frac{1}{s+1}\left(I + \Theta(s)\right), \\ \Phi^u(s) &= \frac{1}{s+1} B_2^{-1}(sI - A)\Theta(s) \;-\; \frac{1}{s+1} B_2^{-1}(A+I), \end{aligned} \tag{2.31}$$

for some $\Theta \in \overline{\mathcal{R}}_s$.

Spatially-Invariant Systems

A spatial Fourier transform (2.2) allows (2.23) to be written equivalently as a *decoupled* family of finite-dimensional systems, indexed by $\lambda \in \hat{\mathbb{G}}$.

$$\hat{x}_\lambda(t+1) \;/\; \frac{d}{dt}\hat{x}_\lambda(t) = \hat{a}_\lambda \hat{x}_\lambda(t) + b_1 \hat{w}_\lambda(t) + b_2 \hat{u}_\lambda(t), \tag{2.32}$$

where $\hat{a}_\lambda = \sum_{n \in \mathbb{G}} a_n e^{-in\lambda}$ is the transform of the convolution kernel defining A, and the finite-dimensional matrices b_1, b_2 define the pointwise multiplication operators B_1, B_2. By Lemma 2.16, the state feedback control

$$\hat{u}_\lambda(z) = \hat{k}_\lambda(z)\hat{x}_\lambda(z)$$

is internally stabilizing for (2.18) if and only if the resulting transfer functions $\hat{\phi}^x_\lambda$ and $\hat{\phi}^u_\lambda$ defined by

$$\begin{aligned} \begin{bmatrix} \hat{x}_\lambda(z) \\ \hat{u}_\lambda(z) \end{bmatrix} &= \begin{bmatrix} \hat{\phi}^x_\lambda(z) \\ \hat{\phi}^x_\lambda(z) \end{bmatrix} b_1 \hat{w}_\lambda(z) \\ &= \begin{bmatrix} (zI - \hat{a}_\lambda - b_2 \hat{k}_\lambda(z))^{-1} \\ \hat{k}_\lambda(z)(zI - \hat{a}_\lambda - b_2 \hat{k}_\lambda(z))^{-1} \end{bmatrix} b_1 \hat{w}_\lambda(z) \end{aligned} \tag{2.33}$$

are in the affine subspace

$$\left. \begin{aligned} &\begin{bmatrix} zI - \hat{a}_\lambda & -b_2 \end{bmatrix} \begin{bmatrix} \hat{\phi}^x_\lambda(z) \\ \hat{\phi}^x_\lambda(z) \end{bmatrix} = I, \\ &\hat{\phi}^x_\lambda,\; \hat{\phi}^u_\lambda \in \overline{\mathcal{R}}_s. \end{aligned} \right\} \tag{2.34}$$

By the results of [16], the state feedback

$$u = Kx,$$

(with K a spatio-temporal system) internally stabilizes the distributed system (2.23) if and only if $\hat{k}_\lambda$ is internally stabilizing for (2.18) for each $\lambda \in \hat{\mathbb{G}}$, where $\hat{k}_\lambda = \sum_{n \in \mathbb{G}} k_n e^{-in\lambda}$ is the Fourier transform of the convolution kernel of K. Thus, the explicit parameterizations provided for finite dimensional systems (2.25), (2.29), (2.31) may be applied to the finite dimensional system (2.18) for each $\lambda \in \hat{\mathbb{G}}$. This allows us to derive a spatially-invariant analogues of Theorems 2.4.1, 2.4.2, 2.4.4.

We begin by considering the discrete-time system (2.23) and in the case that B_2 is an invertible operator, i.e. b_2 is an invertible matrix. Then, the spatially-invariant system K is a stabilizing state-feedback controller for (2.23) if and only if the closed-loop mappings Φ^x and Φ^u are spatially-invariant systems whose convolution kernels $\{\phi_n^x\}_{n\in\mathbb{G}}$ and $\{\phi_n^u\}_{n\in\mathbb{G}}$ are represented in the spatial frequency domain as

$$\begin{aligned} \hat{\phi}_\lambda^x(z) &= \frac{1}{z}\left(I + \hat{\theta}_\lambda(z)\right), \\ \hat{\phi}_\lambda^u(z) &= \frac{1}{z} b_2^{-1}(zI - \hat{a}_\lambda)\hat{\theta}_\lambda - \frac{1}{z} b_2^{-1}\hat{a}_\lambda, \end{aligned} \tag{2.35}$$

for some $\hat{\theta}_\lambda \in \overline{\mathcal{R}}_s$, for all $\lambda \in \hat{\mathbb{G}}$. Taking an inverse spatial Fourier transform, the parameterization (2.35) is written in operator form, as summarized in the following theorem.

Theorem 2.4.5 *The spatially-invariant system K is a stabilizing state-feedback controller for the spatially-invariant discrete-time system (2.23) with B_2 invertible if and only if the closed-loop mappings Φ^x and Φ^u are spatially-invariant systems of the form*

$$\begin{aligned} \Phi^x &= \mathcal{Z}^{-1}\left(I + \Theta\right) \\ \Phi^u &= \mathcal{Z}^{-1} B_2^{-1}\left(\mathcal{Z} - A\right)\Theta \ - \ \mathcal{Z}^{-1} B_2^{-1} A, \end{aligned} \tag{2.36}$$

for some spatially-invariant system $\Theta \in \overline{\mathcal{R}}_s$, where $\mathcal{Z}$ denotes the temporal shift operator, represented in the transfer function domain as multiplication by z.

Similarly, analogous results to Theorems 2.4.4 and 2.4.1 to the spatially-invariant setting are derived as follows.

Theorem 2.4.6 *The spatially-invariant system K is a stabilizing state-feedback controller for the spatially-invariant continuous-time system (2.23) with B_2 invertible if and only if the closed-loop mappings Φ^x and Φ^u are spatially-invariant systems of the form*

$$\begin{aligned} \Phi^x &= (\mathcal{S} + I)^{-1}\left(I + \Theta\right), \\ \Phi^u &= (\mathcal{S} + I)^{-1} B_2^{-1}(\mathcal{S} - A)\Theta - (\mathcal{S} + I)^{-1} B_2^{-1}(A + I), \end{aligned} \tag{2.37}$$

for some spatially-invariant system $\Theta \in \overline{\mathcal{R}}_s$, where $\mathcal{S}$ denotes the temporal differentiation operator, represented in the transfer function domain as multiplication by s.

Theorem 2.4.7 *Consider the continuous (or discrete-time) system (2.23) and assume that $(\mathcal{S} - A)^{-1}$ (or $(\mathcal{Z} - A)^{-1}$) is stable. Then, the spatially-invariant system K is a stabilizing state-feedback controller for (2.23) if and only if the spatially-invariant closed-loop mapping $\Phi^u \in \overline{\mathcal{R}}_s$ and the spatially-invariant system Φ^x is of the form*

$$\Phi^x = \left(\mathcal{Z} - A\right)^{-1}\left(I + B_2\Phi^u\right), \tag{2.38}$$

(or $\Phi^x = \left(\mathcal{S} - A\right)^{-1}\left(I + B_2\Phi^u\right)$).

The explicit parameterizations presented in this section allow for the optimal $\mathcal{H}_2$ controller design problem to be reformulated as a standard model-matching problem. When finite band size constraints on the closed-loops are imposed, this problem has finitely many transfer function parameters, even in the infinite-extent spatially-invariant setting. To illustrate this, we begin by looking at a specific example.

2.4.1 Example

We consider an infinite chain of first-order discrete-time subsystems with nearest neighbor interactions and fully distributed control

$$x_n(t+1) = \alpha\Big(x_n(t) + \kappa x_{n+1}(t) + \kappa x_{n-1}(t)\Big) + w_n(t) + u_n(t). \tag{2.39}$$

The parameters α and κ determine stability and coupling strength between the subsystems respectively. (We will later generalize this model beyond the nearest neighbor setting.) We write (2.39) in operator notation as:

$$x \;=\; (\mathcal{Z}I - A)^{-1}\,(w+u) \;=\; \begin{bmatrix}\mathcal{P}_{21} & \mathcal{P}_{22}\end{bmatrix}\begin{bmatrix} w \\ u\end{bmatrix}. \tag{2.40}$$

Here $\mathcal{Z}$ is the temporal left-shift operator and A is the operator of (discrete) spatial convolution by the sequence $\{a_n\} = \{\cdots, 0, 0, \kappa\alpha, \alpha, \kappa\alpha, 0, 0, \cdots\}$, i.e.

$$a_n := \begin{cases} \kappa\alpha, & |n| = 1 \\ \alpha, & n = 0 \\ 0, & |n| > 1. \end{cases}$$

We consider the set of stabilizing, dynamic state feedback controllers K

$$u \;=\; K\,x,$$

and optimize over the closed-loop mappings Φ^x and Φ^u

$$\begin{bmatrix} x \\ u \end{bmatrix} \;=\; \begin{bmatrix}\Phi^x \\ \Phi^u\end{bmatrix} w \;:=\; \begin{bmatrix} I \\ K\end{bmatrix}(I - \mathcal{P}_{22}K)^{-1}\,\mathcal{P}_{21}\;w.$$

We precisely state the optimal control problem of interest:
Optimal $\mathcal{H}_2$ design with prescribed closed-loop band size:

$$\begin{aligned} J_N := \inf_{K \text{ stabilizing}} \quad & \|\Phi^x\|_2^2 \;+\; \gamma\,\|\Phi^u\|_2^2 \\ \text{s.t.} \quad & \Phi^x \text{ has spatial sparsity extent } N \\ & K \text{ spatially invariant.} \end{aligned} \tag{2.41}$$

Here γ quantifies the usual tradeoff between disturbance attenuation and control effort. Constraining K to be spatially invariant guarantees that the closed-loop maps Φ^x and Φ^u will also be spatially invariant, so the objective function

$$\|\Phi^x\|_2^2 \;+\; \gamma\;\|\Phi^u\|_2^2$$

is well-defined. Without the spatial sparsity constraint, the above problem is an infinite-dimensional $\mathcal{H}_2$ problem for spatially invariant systems which can be solved using the techniques of [16]. Imposing this constraint makes the problem different and has the following consequences, which we will demonstrate:

1. If Φ^x has spatial sparsity extent N, then Φ^u has spatial sparsity extent $(N+1)$.
2. If Φ^x has spatial sparsity extent N, then we can construct a dynamic controller with spatial extent $(N+1)$.
3. The above problem can be converted to a standard finite-dimensional $\mathcal{H}_2$ *model-matching* problem with $(2N+1)$ transfer function parameters.

Although a problem of interest is to impose spatial sparsity constraints directly on K, this problem is in general not convex. As argued in [11], an advantage of the problem formulation (2.41) is that it is a convex problem. Remark 2 above shows that the two problems are close in some sense since the latter yields a controller with spatial sparsity extent $(N+1)$, only one more than that prescribed for the closed-loop mapping Φ^x.

We show that optimization problem (2.41) reduces to a finite dimensional $\mathcal{H}_2$ model matching problem whose size depends on the spatial sparsity extent constraint on the closed-loop system responses. We first introduce some notation which will be used to analyze this example. Let $\{\phi_n^x(z)\}$ be the convolution kernel defining the spatially invariant system Φ^x in the transfer function domain, and suppose Φ^x has spatial sparsity of extent N. We define

$$r_n(z) := \begin{cases} z^2 \phi_n^x(z)\ , & 0 < |n| \leq N \\ z^2\left(\phi_n^x(z) + z^{-1}\right)\ , & n = 0 \\ 0\ , & \text{else} \end{cases}$$

and let R denote the transfer matrix of size $(2N+1)\times 1$ constructed by:

$$\overline{R} := \begin{bmatrix} r_{-N} \\ \vdots \\ r_0 \\ \vdots \\ r_N \end{bmatrix}. \tag{2.42}$$

Theorem 2.4.8 *The constrained optimization problem* (2.41) *is equivalent to the following finite dimensional* $\mathcal{H}_2$ *problem:*

$$\begin{aligned} J_N = \quad & \inf_{\overline{R}} \quad \|H + VR\|_2^2 \\ & s.t. \quad R_n \in \overline{\mathcal{R}}_s \text{ for all } |n| \leq N, \end{aligned} \tag{2.43}$$

where H *and* V *are finite-dimensional transfer matrices constructed from the plant using the following procedure:*

$$H(z) = \begin{bmatrix} H_1(z) \\ H_2(z) \\ \vdots \\ H_{4N+4}(z) \end{bmatrix}$$

is a transfer matrix of size $(4N+4) \times 1$*, whose entries are given by*

$$H_n(z) := \begin{cases} \frac{1}{z} \,, & n = N+1 \\ \frac{-\alpha\kappa\sqrt{\gamma}}{z} \,, & n = 3N+2, 3N+4 \\ \frac{-\alpha\sqrt{\gamma}}{z} \,, & n = 3N+3 \\ 0 \,, & \text{else,} \end{cases}$$

V *is defined as the following block matrix:*

$$V := \begin{bmatrix} \frac{1}{z^2} I_{2N+1} \\ \frac{1}{z^2} V_L \end{bmatrix}$$

where I_{2N+1} *denotes the identity matrix of size* $(2N+1) \times (2N+1)$*, and* V_L *is the lower triangular transfer matrix of size* $(2N+3) \times (2N+1)$ *defined by*

$$V_L := \sqrt{\gamma} \cdot \begin{bmatrix} -\alpha\kappa & 0 & 0 & \cdots \\ z-\alpha & -\alpha\kappa & 0 & \cdots \\ -\alpha\kappa & z-\alpha & -\alpha\kappa & \cdots \\ 0 & \ddots & \ddots & \ddots \end{bmatrix}.$$

Note that there are $2N+1$ parameters (transfer functions) in optimization problem (2.43), where N is the spatial sparsity extent imposed on Φ^x.

We prove Theorem 2.4.8 through a series of lemmas. We begin by relating the spatial sparsity extent of Φ^x to that of Φ^u and writing Φ^u as a function of Φ^x.

Lemma 2.4.9 *Let* Φ^x *and* Φ^u *satisfy constraint* (2.17)*, and assume that* Φ^x *has spatial sparsity extent* N*. Then* Φ^u *has spatial sparsity extent* $N+1$*. The non-zero entries of the transfer function convolution kernels defining the spatially-invariant systems* Φ^x *and* Φ^u *are related by*

$$\begin{bmatrix} \phi^u_{-N-1}(z) \\ \cdots \\ \phi^u_{N+1}(z) \end{bmatrix} = V_L \begin{bmatrix} \phi^x_{-N}(z) \\ \cdots \\ \phi^x_{N}(z) \end{bmatrix} \tag{2.44}$$

where V_L *is defined as in Theorem 2.4.8.*

Proof: If Φ^x has band size N then $\hat{\Phi}^x$ (the spatial Fourier transform of Φ^x) has the following representation:

$$\hat{\Phi}^x_\lambda(z) = \sum_{n=-N}^{N} \phi^x_n(z) e^{-in\lambda}. \tag{2.45}$$

If Φ^x and Φ^u also satisfy (2.17), then (**??**) holds and can be used to show that:

$$\hat{\Phi}^u_\lambda(z) = \Big(-\kappa\alpha \left(e^{i\lambda} + e^{-i\lambda}\right) + (z-\alpha)\Big)\hat{\Phi}^x_\lambda(z) - 1 \;=\; \sum_{n=-N-1}^{N+1} \phi^u_n(z) e^{-in\lambda}, \tag{2.46}$$

demonstrating that Φ^u has spatial sparsity extent $(N+1)$. Equation (2.44) then follows from (2.46). ■

This allows us to write the constraints in optimization problem (2.41) strictly in terms of the transfer function parameter $\overline{R}$ (2.42).

Lemma 2.4.10 *Given that Φ^x has spatial sparsity extent N, the constraint $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ in optimization problem* (2.41) *is equivalent to requiring $R \in \overline{\mathcal{R}}_s$ where R is given by* (2.42)*.*

Proof: Suppose Φ^x, $\Phi^u \in \overline{\mathcal{R}}_s$. Then $\hat{\Phi}^u_\lambda$ is strictly proper for all λ, i.e. $\left(z\hat{\Phi}^x_\lambda + 1\right) \to 0$ as $|z| \to \infty$. Writing

$$1 + z\hat{\Phi}^x_\lambda(z) = 1 + \sum_{n=-N}^{N} \phi^x_n(1) e^{-in\lambda} + z^{-1} \left(\sum_{n=-N}^{N} \sum_{t=1}^{\infty} \phi^x_n(t) e^{-in\lambda} z^{-t+1} \right), \tag{2.47}$$

we see that $1 + \sum_{n=-N}^{N} \phi^x_n(1) e^{-in\lambda} = 0$ for all λ. From this it can be shown that r_n is proper for all n. It is straightforward to check that each r_n must also be stable. The converse follows similarly. ■

$\mathcal{H}_2$ problem

Lemma 2.4.9 allows us to write the objective function in terms of $\overline{R}$ as:

$$\begin{aligned} \|\Phi^x\|_2^2 + \gamma\|\Phi^u\|_2^2 &= \left\| \begin{bmatrix} \phi^x_{-N} & \cdots \phi^x_N & \sqrt{\gamma}\phi^u_{-N-1} & \cdots & \sqrt{\gamma}\phi^u_{N+1} \end{bmatrix}^T \right\|_2^2 \\ &= \|H + VR\|_2^2, \end{aligned}$$

where H and V are as defined in Theorem 2.4.8. Combining this with the results of Lemma 2.4.10, optimization problem (2.41) is equivalent to:

$$\begin{aligned} J_N = \;& \inf_R \quad \|H + VR\|_2^2 \\ & s.t. \quad R \in \overline{\mathcal{R}}_s, \end{aligned}$$

completing the proof of Theorem 2.4.8. ■

Note that the optimal transfer function parameter R for problem (2.43) may be used to construct the optimal spatially-invariant systems Φ^x and Φ^u, and these mappings can be used to directly implement the optimal control policy $K = \Phi^u(\Phi^x)^{-1}$.

It is straightforward to generalize these results to a spatially invariant system with dynamics coupled between a general number of neighbors, rather than just nearest neighbor interactions:

$$x_n(t+1) = \alpha x_n(t) + \alpha\kappa_1 \left(x_{n+1}(t) + x_{n-1}(t)\right) + \alpha\kappa_p \left(x_{n+p}(t) + x_{n-p}(t)\right) + u_n(t) + w_n(t). \tag{2.48}$$

Following the same steps as for the case of $p = 1$, we see that if Φ^x has spatial sparsity of extent N, then Φ^u has spatial sparsity of extent $N + p$, and in this case (2.41) can be reduced to a finite dimensional $\mathcal{H}_2$ model matching problem with $(2N + p)$ parameters.

Numerical Example

We solve (2.43) for the case of $\kappa = 0.8$, $\gamma = 1$ and $\alpha = 1.5$ and allow the spatial sparsity extent constraint N to vary. For comparison, we use the methods presented in [16] to numerically calculate the cost associated with the optimal unconstrained state feedback controller, denoted as J^*.

The optimal constrained cost J_N is plotted as a function of N in Figure 2.1, demonstrating that J_N converges to the unconstrained cost J^* (plotted with a red line) as $N \to \infty$. Furthermore, the semilog plot of $J_N - J^*$ is almost linear, indicating that the convergence is exponential. It is an interesting question as to whether this exponential convergence holds true not only for this example, but is a general property.

We comment on the interpretation of the decrease of J_N with increasing N. Recall that J_N is a mixture of disturbance attenuation and control effort (2.41). Demanding that Φ^x has spatial sparsity extent N means that the effect of a disturbance on the state must be reduced to zero at all locations further than N steps away, and at all times. This is a fairly stringent requirement, and the price paid for it is that the optimal controller will have relatively higher values of J_N. This means that when the impulse disturbance is quenched at locations further than N neighbors, there will be a higher value of the ℓ^2 response norm within the N neighbors. In other words, it appears that *a controller designed to "isolate" neighbors further than N steps away from disturbance effects will necessarily expose neighbors within N steps to worse effects of same disturbance*. Again, it will be interesting to explore whether this phenomenon is peculiar to this example or if it holds in general.

2.4.2 Optimal Controller Design with Closed-Loop Parameterizations

Methods similar to those employed in Section 2.4.1 to convert the specific problem (2.39) to a standard model-matching problem with finitely many transfer function pa-

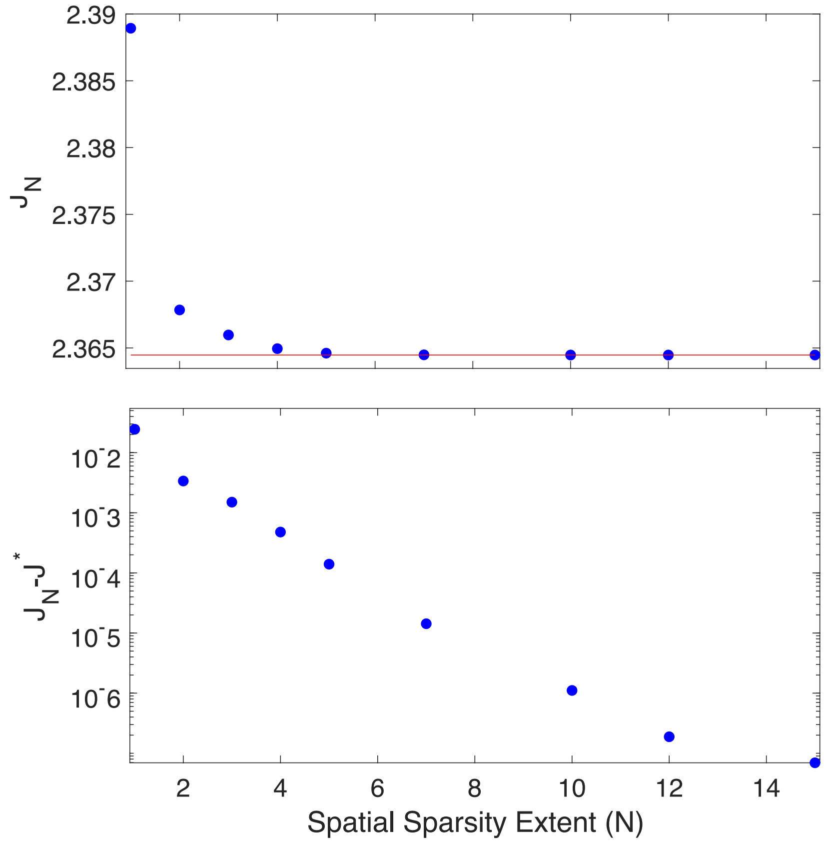


Figure 2.1: *(Top)* A plot of the optimal performance J_N with spatial sparsity constraint N. As expected, J_N limits to J^*(plotted with a red line), the optimal cost with no constraints. A semilog plot (*Bottom*) indicates that this convergence is exponential.

rameters can be applied more generally as well. For instance, consider a discrete-time finite-dimensional system (2.23) with B_2 invertible so that parameterization (2.29) can be applied. Let

$$\overline{z}(t) = C_1 x(t) + D_{12} u(t),$$

denote the performance output of interest. The unconstrained problem which optimizes the $\mathcal{H}_2$ norm to performance output $\overline{z}$:

$$\begin{aligned} \inf_{K} \quad & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} \quad & K \text{ stabilizing}, \end{aligned} \tag{2.49}$$

can be written in terms of the parameter Θ using parameterization (2.29):

$$\begin{aligned} &\inf_{\Theta} \quad \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \frac{1}{z}I + \frac{1}{z}\Theta \\ -\frac{1}{z}B_2^{-1}A + \frac{1}{z}B_2^{-1}(zI - A)\Theta \end{bmatrix} B_1 \right\|_{\mathcal{H}_2}^2 \\ &\text{s.t.} \quad \Theta \in \overline{\mathcal{R}}_s \\ = &\inf_{\Theta \in \overline{\mathcal{R}}_s} \left\| \tfrac{1}{z}\left(C_1 - D_{12}B_2^{-1}A\right) B_1 \; + \; \tfrac{1}{z}\left(C_1 + D_{12}B_2^{-1}(zI - A)\right) \; \Theta \; B_1 \right\|_{\mathcal{H}_2}^2, \end{aligned} \tag{2.50}$$

which is in the form of a standard model-matching problem.

An analogous formulation holds in the infinite extent spatially-invariant setting as well, when C_1 and D_{12} are pointwise multiplication operators. In this case, band size constraints on the closed-loops can be written in terms of Θ, noting that the band size of Θ and Φ^x will be the same and the band size of Φ^u will be L greater than that of Θ, where L is the band size of the spatial convolution operator A, e.g.

$$\begin{aligned} &\begin{cases} \inf_{K} & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} B_1 \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & K \text{ stabilizing, spatially-invariant} \\ & \Phi^x \text{ has band size } M, \; \Phi^u \text{ has band size } (M + L) \end{cases} \\ = &\begin{cases} \inf_{\Theta \in \overline{\mathcal{R}}_s} & \left\| \tfrac{1}{z}\left(C_1 - D_{12}B_2^{-1}A\right) B_1 \; + \; \tfrac{1}{z}\left(C_1 + D_{12}B_2^{-1}(zI - A)\right) \; \Theta \; B_1 \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & \Theta \text{ spatially-invariant with band size } M. \end{cases} \end{aligned} \tag{2.51}$$

Following a similar procedure as in Section 2.4.1, (2.51) can be converted to a standard finite-dimensional model-matching problem with $(2M + 1)$ transfer function parameters, where M is the band size constraint on Θ. A similar formulation holds in the continuous-time setting as well.

2.5 Part 2: Decoupled Subsystem Dynamics

Throughout part 2 of this chapter, we make the following assumptions:

1. **Decoupled Subsystem Dynamics:** The state equation of (2.11) can be written as:
$$\dot{x}_n(t) = A^{(n)}x_n(t) + B_1^{(n)}w_n(t) + B_2^{(n)}u_n(t), \; n \in \mathbb{G}, \tag{2.52}$$
i.e. the dynamics of subsystem n are independent of all other subsystems $m \neq n$.

We begin by additionally assuming:

2. **Spatial Invariance:** C_1 and D_{12} are spatial convolution operators, and A, B_1, B_2 are pointwise multiplication operators so that $A^{(n)} = A, \; B_1^{(n)} = B_1, \; B_2^{(n)} = B_2$ are independent of n.

In Section 2.5.7, we will analyze *spatially-varying* systems, which satisfy the decoupled dynamics condition (1) but may not satisfy the spatial invariance condition (2).

2.5.1 Motivating Applications

We consider the following two applications:

- **Consensus of first order subsystems:** The dynamics are given by

$$\dot{x}_n = u_n + w_n, \quad n \in \mathbb{G}, \tag{2.53}$$

with x_n the scalar-valued state at spatial site n. Applications for the first order consensus problem include load balancing over a distributed file system.

- **Vehicular platoons (consensus of second order subsystems):** Following [3], we let ξ_n represent the absolute deviation of vehicle n from a desired trajectory $\overline{\xi}_n$

$$\overline{\xi}_n := \overline{v}t + n\delta,$$

with $\overline{v}$ the specified cruising velocity. Defining $v_n := \dot{\xi}_n$, the dynamics are given by

$$\dot{x}_n = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} x_n + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_n + \begin{bmatrix} 1 \\ 0 \end{bmatrix} w_n, \; n \in \mathbb{G}, \tag{2.54}$$

where $x_n := \begin{bmatrix} (v_n + \xi_n) & \xi_n \end{bmatrix}^T$.

For an appropriately chosen performance output $\overline{z}$, the generalized plant P which describes either of these systems is spatially-invariant and of the form (2.11). The $\mathcal{H}_2$ design problem for either of these systems, subject to constraints on the closed-loop band size, is then of the form (2.15).

2.5.2 Closed-Loop Parametrization: Locally 1$^{\text{st}}$ Order Systems

In this section, we study systems with dynamics of the form:

$$\dot{x}_n = a x_n + w_n + u_n, \; n \in \mathbb{G}. \tag{2.55}$$

When the system (2.55) is written in vector form

$$\dot{x}(t) \;=\; Ax(t) \;+\; w(t) \;+\; u(t), \tag{2.56}$$

the "A-matrix", $A := aI$, is a multiple of the (possibly infinite extent) identity matrix. At each spatial site, the state x_n is first order. The dimension of the overall state vector $x(.,.)$ is equal to the cardinality of $\mathbb{G}$, and can be either finite or infinite. We call this class of systems *spatially-invariant locally* 1^{st} *order.* The following theorem provides an explicit parameterization of all achievable stabilized closed-loop mappings for spatially-invariant locally 1$^{\text{st}}$ order systems of the form (2.55), in the case that $\Re(a) \geq 0$.

Theorem 2.5.1 *Let K be a spatially-invariant system. Then the controller $u = Kx$ is stabilizing for (2.55), with $\Re(a) \geq 0$, if and only if the resulting closed-loop mappings are of the form*

$$\Phi^x = (sI + I)^{-1}\Theta + (sI + I)^{-1} \tag{2.57}$$

$$\Phi^u = (s-a)(sI + I)^{-1}\Theta - (a+1)(sI + I)^{-1} \tag{2.58}$$

for some spatially-invariant system $\Theta = \{\theta_n(s)\}_{n\in\mathbb{G}} \in \overline{\mathcal{R}}_s$.

Proof: We consider first the finite space setting, $\mathbb{G} = \mathbb{Z}_N$. Assume K is stabilizing. Then, following the results of [11], the definitions (2.13) of Φ^x and Φ^u show that

$$(sI - A)\Phi^x(s) \ - \Phi^u(s) \ = \ I, \tag{2.59}$$

and $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$. Using the fact that $(sI - A)^{-1} = (sI - aI)^{-1}$, the affine relation (2.59) can be stated as

$$\Phi^x(s) = \frac{1}{s-a}\left(I + \Phi^u(s)\right). \tag{2.60}$$

Then $I + \Phi^u(s)$ must have a zero at a to ensure that Φ^x defined by (2.60) does note have an unstable pole at a, i.e. Φ^u must be of the form

$$\Phi^u(s) = \frac{s-a}{s+1}\Theta(s) - \frac{a+1}{s+1}I,$$

for some $\Theta \in \overline{\mathcal{R}}_s$. Substituting this expression for Φ^u into (2.60) shows Φ^x is of the form

$$\Phi^x(s) = \frac{1}{s+1}\Theta(s) - \frac{1}{s+1}I.$$

Conversely, if the finite-dimensional transfer functions Φ^x and Φ^u are of the form (2.57), (2.58), then $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ so that K is stabilizing. Details for the infinite space setting are presented in Appendix 2.7.1.

■

We note that in the case that $\Re(a) < 0$, a parameterization of all achievable stabilized closed-loop mappings for (2.55) is provided in Part 1 of this Chapter (Section 2.4).

2.5.3 Locally n^{th} Order Systems: A Back-Stepping Approach

We next generalize the results of Section 2.5.2 to the case of subsystems with higher order dynamics, using a backstepping-like procedure. We begin by considering a single finite-dimensional system with dynamics of the form

$$\dot{x} = Ax + B_1 w + B_2 u, \tag{2.61}$$

with (A, B_2) controllable. We assume that $(A+I, B_2)$ is in *controllable-canonical form* [27], i.e.

$$
(A+I) = \begin{bmatrix} -a_1 I & -a_2 I & -a_3 I & \cdots & -a_n I \\ I & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & I & 0 \end{bmatrix}, \qquad (2.62)
$$
$$
B_2 = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \end{bmatrix}^T,
$$

where n is defined to be the order of the system. We note that this assumed form is somewhat nontraditional, and is chosen to simplify the use of Lemma 2.4.3. This assumption is without loss of generality, as demonstrated by the following Proposition.

Proposition 2.5.2 *If the finite dimensional system (A, B_2) is controllable, then there exists an invertible transformation matrix T such that $\hat{A} := TAT^{-1}, \hat{B}_2 := TB_2$ and $(\hat{A}+I, \hat{B}_2)$ is in controllable-canonical form, i.e. of the form* (2.62)*.*

Proof: A and $(A+I)$ have the same set of eigenvectors, so that (A, B_2) is controllable if and only if $(A+I, B_2)$ is. Since $(A+I, B_2)$ is controllable, there exists a similarity transformation which converts the system to controllable canonical form [27]. ■

To extend the results of Section 2.5.2 to the case of higher order subsystems, we employ Lemma 2.4.3. The affine subspace modification provided by this Lemma is useful as the operation of multiplication by $(sI+I)$ has an inverse that preserves stability, i.e. $(sI+I)^{-1}\Theta \in \overline{\mathcal{R}}_s$ whenever $\Theta \in \overline{\mathcal{R}}_s$. We remark that Lemma 2.4.3 was similarly employed to prove Theorem 2.4.4 in Part 1 of this chapter.

We employ Lemma 2.4.3 to prove the following theorem, which explicitly parameterizes the set of all achievable stabilized closed-loop mappings for (2.61).

Theorem 2.5.3 *A (dynamic or static) controller $u = Kx$ is stabilizing for* (2.61)*, with $(A+I, B_2)$ in controllable-canonical form, if and only if the resulting closed-loop mappings are of the form*

$$
\Phi^x(s) = \begin{bmatrix} \frac{1}{s+1} I \\ \frac{1}{(s+1)^2} I \\ \vdots \\ \frac{1}{(s+1)^n} I \end{bmatrix} \begin{bmatrix} \theta_1(s) & \cdots & \theta_n(s) \end{bmatrix} + \begin{bmatrix} \frac{1}{s+1} I & 0 & 0 & \cdots & 0 \\ \frac{1}{(s+1)^2} I & \frac{1}{s+1} I & 0 & \cdots & 0 \\ \frac{1}{(s+1)^3} I & \frac{1}{(s+1)^2} I & \frac{1}{s+1} I & \cdots & 0 \\ \vdots & & & \ddots & \\ \frac{1}{(s+1)^n} I & \frac{1}{(s+1)^{n-1}} I & \frac{1}{(s+1)^{n-2}} I & \cdots & \frac{1}{s+1} I \end{bmatrix}
$$
$$
=: F(s) \begin{bmatrix} \theta_1(s) & \cdots & \theta_n(s) \end{bmatrix} + L(s),
$$
$$
\Phi^u(s) = \chi(s) \begin{bmatrix} \theta_1(s) & \cdots & \theta_n(s) \end{bmatrix} + \eta(s) \qquad (2.63)
$$

for some $\theta = \begin{bmatrix} \theta_1 & \cdots & \theta_n \end{bmatrix} \in \overline{\mathcal{R}}_s$, *where*

$$\chi(s) := 1 + \frac{a_1}{s+1} + \frac{a_2}{(s+1)^2} + \cdots + \frac{a_n}{(s+1)^n}$$

with the a_i*'s given by* (2.62) *and*

$$\eta(s) = \begin{bmatrix} \eta_1(s) & \eta_2(s) & \cdots & \eta_n(s) \end{bmatrix}, \quad \eta_k(s) := \sum_{i=k}^{n} \frac{a_i}{(s+1)^{i+1-k}} I.$$

Proof: By Lemma 2.4.3, it is sufficient to show that $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ satisfy the affine constraint (2.30) if and only if they are of the form (2.63). The proof of this follows from a back-stepping procedure similar to the back-stepping approach for strict feedback systems presented in [28]. We note that a variety of works have employed similar back-stepping techniques, e.g. [29, 30].

If Φ^x, Φ^u are of the form (2.63) for some $\Theta \in \overline{\mathcal{R}}_s$, then $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ and direct computations show that these mappings satisfy (2.30). Conversely, assume $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ satisfy (2.30). The following back-stepping procedure demonstrates that Φ^x, Φ^u are of the form (2.63).

Backstepping Algorithm

Partition Φ^x by block rows as

$$\Phi^x = \begin{bmatrix} \Phi^x_{11} & \Phi^x_{12} & \cdots & \Phi^x_{1n} \\ \Phi^x_{21} & \Phi^x_{22} & \cdots & \Phi^x_{2n} \\ \vdots & & & \\ \Phi^x_{n1} & \Phi^x_{n2} & \cdots & \Phi^x_{nn} \end{bmatrix} =: \begin{bmatrix} \Phi^x_1 \\ \Phi^x_2 \\ \vdots \\ \Phi^x_n \end{bmatrix}$$

If $(A + I)$ and B_2 are of the form (2.61), then

$$((s+1)\,I - (A+I)) = \begin{bmatrix} (s+1+a_1)I & a_2 I & \cdots & a_{n-1}I & a_n I \\ -I & (s+1)I & \cdots & 0 & 0 \\ 0 & -I & \cdots & 0 & 0 \\ \vdots & & \ddots & & \\ 0 & 0 & \cdots & -I & (s+1)I \end{bmatrix},$$

and for any $k \neq 1$, the k^{th} block row of the affine constraint (2.30) can be rearranged as:

$$\Phi^x_k = \frac{1}{(s+1)} \left(E_k + \Phi^x_{k-1} \right), \tag{2.64}$$

where E_k is defined to be the matrix whose k^{th} block entry is the identity, and all other entries are zeros:

$$E_k := \begin{bmatrix} 0 & \cdots & 0 & I & 0 & \cdots & 0 \end{bmatrix}.$$

Back-substituting, each block row, Φ^x_k, can be written in terms of the first block row, Φ^x_1, as:

$$\begin{aligned}\Phi^x_k = &\frac{1}{(s+1)^{k-1}}\Phi^x_1 + \frac{1}{s+1}E_k + \frac{1}{(s+1)^2}E_{k-1} \\ &+ \frac{1}{(s+1)^{k-1}}E_2\end{aligned} \tag{2.65}$$

From (2.65) it follows that $\Phi^x \in \overline{\mathcal{R}}_s$ whenever $\Phi^x_1 \in \overline{\mathcal{R}}_s$. Rearranging the first block row, Φ^x_1, of (2.30) shows that

$$\Phi^u = (s+1+a_1)\Phi^x_1 - E_1 - \sum_{k=2}^{n} \Phi^x_k \tag{2.66}$$

Substituting (2.65) into (2.66) shows that:

$$\Phi^u_k = \begin{cases} \alpha(s)\Phi^x_{1k}(s) + \sum_{i=k}^{n} \frac{a_i}{(s+1)^{i+1-k}}I, \ k \neq 1 \\ \alpha(s)\Phi^x_{11}(s) - I, \ k = 1 \end{cases} \tag{2.67}$$

where $\alpha(s) := (s+1) + a_1 + \frac{a_2}{s+1} + \cdots + \frac{a_n}{(s+1)^{n-1}}$, and we have partitioned Φ^u and each Φ^x_k by block columns as:

$$\Phi^u = \begin{bmatrix} \Phi^u_1 & \Phi^u_2 \cdots & \Phi^u_n \end{bmatrix}$$
$$\Phi^x_k = \begin{bmatrix} \Phi^x_{k1} & \Phi^x_{k2} & \cdots & \Phi^x_{kn} \end{bmatrix}$$

From (2.67), we see that if $\Phi^x_{1k} \in \overline{\mathcal{R}}_s$, then the following is a necessary and sufficient condition for $\Phi^u_k \in \overline{\mathcal{R}}_s$:

$$\begin{cases} (s+1)\Phi^x_{1k} \in \overline{\mathcal{R}}_s, \quad k \neq 1, \\ (s+1)\Phi^x_{11}(s) - I \in \overline{\mathcal{R}}_s, \quad k = 1. \end{cases}$$

Equivalently,

$$\Phi^x_{1k}(s) = \begin{cases} \frac{1}{s+1}\theta_k(s), \ k \neq 1 \\ \frac{1}{s+1}\theta_k(s) + \frac{1}{s+1}I, \ k = 1 \end{cases} \tag{2.68}$$

for some $\theta_k \in \overline{\mathcal{R}}_s$. Substituting (2.68) into (2.67) and (2.65) shows that Φ^u and Φ^x are of the form (2.63). ■

Remark 1 *Equation* (2.63) *of Theorem 2.5.3 can be modified to parameterize all stabilized closed loops for discrete-time systems:*

$$x(t+1) = Ax(t) + B_1w(t) + B_2u(t),$$

with (A, B_2) *in* standard *controllable-canonical form [27], by replacing all* $(s+1)^{-1}$ *terms with* z^{-1} *and using the discrete-time definitions of* Φ^x, Φ^u *presented in [11].*

We next use the result of Theorem 2.5.3 to provide an explicit parameterization of all stabilized closed-loop maps for a spatially-invariant system composed of subsystems that each have dynamics of the form (2.61).

Spatially-Invariant Locally n^{th} Order Systems

Consider a spatially-invariant system (of finite or infinite spatial extent) with dynamics of the form

$$\dot{x}_m = Ax_m + B_1 w_m + B_2 u_m,\ m \in \mathbb{G}, \tag{2.69}$$

with (A, B_2) controllable so that, without loss of generality, $(A+I, B_2)$ is in controllable-canonical form. We let n denote the order of each subsystem so that

$$x_m = \begin{bmatrix} x_{m1}^T & x_{m2}^T & \cdots & x_{mn}^T \end{bmatrix}^T$$

for each m, and refer to systems of the form (2.69) as *spatially-invariant locally n^{th} order.*

Theorem 2.5.4 *Let K be a spatially-invariant system. The controller $u = Kx$ is stabilizing for (2.69) if and only if the resulting closed-loop mappings $\Phi^x = \{\phi_m^x\}_{m\in\mathbb{G}}$ and $\Phi^u = \{\phi_m^u\}_{m\in\mathbb{G}}$ are of the form:*

$$\begin{aligned} \phi_m^x(s) &= \begin{cases} F(s)\theta_m(s) + L(s),\ m = 0 \\ F(s)\theta_m(s),\ m \neq 0 \end{cases} \\ \phi_m^u(s) &= \begin{cases} \chi(s)\theta_m(s) + \eta(s),\ m = 0 \\ \chi(s)\theta_m(s),\ m \neq 0, \end{cases} \end{aligned} \tag{2.70}$$

for some spatially-invariant system $\Theta = \{\theta_m\}_{m\in\mathbb{G}} \in \overline{\mathcal{R}}_s$, where $F(s), L(s), \chi(s)$, and $\eta(s)$ are defined as in Theorem 2.5.3. Equivalently,

$$\begin{aligned} \Phi^x &= F\Theta + L \\ \Phi^u &= \chi\Theta + \eta, \end{aligned} \tag{2.71}$$

where F, L, χ, η are spatially-invariant systems defined by pointwise multiplication by $F(s), L(s), \chi(s), \eta(s)$.

A proof of this result is provided in Appendix 2.7.2

Theorem 2.5.4 complements the results of Part 1 of this chapter (Section 2.4). This theorem applies to the general class of decoupled subsystems for which (A, B_2) is controllable. In particular, this result may be applied to the vehicular platoons problem, which has both B_2 not invertible and is unstable in open-loop. This is in contrast to the parameterizations provided in Part 1 of this chapter, which do not apply to the case that B_2 is not invertible and require additional analysis in the case that the open-loop is unstable. Unlike Theorem 2.5.4, however, recall that the parameterizations of Part 1 may be applied to the case of subsystems with *coupled dynamics*.

This parameterization, along with the parameterizations introduced in Part 1 of this section, are summarized for both discrete and continuous-time settings in the following tables. In the discrete-time, decoupled open-loop setting (row 3 of table) we assume (A, B_2) is in controllable canonical form:

$$A = \begin{bmatrix} -a_1 I & -a_2 I & -a_3 I & \cdots & -a_n I \\ I & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \ddots & \\ 0 & & & I & 0 \end{bmatrix}, \; B_2 = \begin{bmatrix} I \\ 0 \\ 0 \\ \cdots \\ 0 \end{bmatrix}.$$

and in the continuous-time, decoupled open-loop setting (row 3 of table) we assume $(A + I, B_2)$ is in controllable canonical form (2.62).

Discrete-Time

	Closed-Loop Parameterization
Stable Open-Loop	$\Phi^x(z) = (zI - A)^{-1}(I + B_2\Phi^u(z)), \quad \Phi^u \in \overline{\mathcal{R}}_s$
B_2 Invertible	$\Phi^x(z) = \frac{1}{z}(I + \Theta(z)),$ $\Phi^u(z) = \frac{1}{z}B_2^{-1}\left((zI - A)\Theta(z) - A\right), \quad \Theta \in \overline{\mathcal{R}}_s$
Decoupled Open-Loop	$\Phi^x(z) = F(z)\Theta(z) + L(z)$ $\Phi^u(z) = \chi(z)\Theta(z) + \eta(z), \quad \Theta \in \overline{\mathcal{R}}_s$

$$\text{where } F(z) = \begin{bmatrix} \frac{1}{z}I \\ \frac{1}{z^2}I \\ \vdots \\ \frac{1}{z^n}I \end{bmatrix}, \; L(z) = \begin{bmatrix} \frac{1}{z}I & 0 & 0 & \cdots & 0 \\ \frac{1}{z^2}I & \frac{1}{z}I & 0 & \cdots & 0 \\ \frac{1}{z^3}I & \frac{1}{z^2}I & \frac{1}{s+1}I & \cdots & 0 \\ \vdots & & & \ddots & \\ \frac{1}{z^n}I & \frac{1}{z^{n-1}}I & \frac{1}{z^{n-2}}I & \cdots & \frac{1}{z}I \end{bmatrix},$$

$\chi(z) := 1 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots + \frac{a_n}{z^n}$, and $\eta(z) = \begin{bmatrix} \eta_1(z) & \eta_2(z) & \cdots & \eta_n(z) \end{bmatrix}$, $\eta_k(z) := \sum_{i=k}^{n} \frac{a_i}{z^{i+1-k}} I$.

Continuous-Time

	Closed-Loop Parameterization
Stable Open-Loop	$\Phi^x(s) = (sI - A)^{-1}(I + B_2\Phi^u(s)), \quad \Phi^u \in \overline{\mathcal{R}}_s$
B_2 Invertible	$\Phi^x(s) = \frac{1}{s+1}\left(I + \Theta(s)\right),$ $\Phi^u(s) = \frac{1}{s+1}B_2^{-1}\left((sI - A)\Theta(s) - (A + I)\right), \quad \Theta \in \overline{\mathcal{R}}_s$
Decoupled Open-Loop	$\Phi^x(s) = F(s)\Theta(s) + L(s)$ $\Phi^u(s) = \chi(s)\Theta(s) + \eta(s), \quad \Theta \in \overline{\mathcal{R}}_s$

where $F(s) = \begin{bmatrix} \frac{1}{s+1}I \\ \frac{1}{(s+1)^2}I \\ \vdots \\ \frac{1}{(s+1)^n}I \end{bmatrix}$, $L(s) = \begin{bmatrix} \frac{1}{s+1}I & 0 & 0 & \cdots & 0 \\ \frac{1}{(s+1)^2}I & \frac{1}{s+1}I & 0 & \cdots & 0 \\ \frac{1}{(s+1)^3}I & \frac{1}{(s+1)^2}I & \frac{1}{s+1}I & \cdots & 0 \\ \vdots & & & \ddots & \\ \frac{1}{(s+1)^n}I & \frac{1}{(s+1)^{n-1}}I & \frac{1}{(s+1)^{n-2}}I & \cdots & \frac{1}{s+1}I \end{bmatrix}$,
$\chi(s) := 1 + \frac{a_1}{s+1} + \frac{a_2}{(s+1)^2} + \cdots + \frac{a_n}{(s+1)^n}$, and $\eta(s) = \begin{bmatrix} \eta_1(s) & \eta_2(s) & \cdots & \eta_n(s) \end{bmatrix}$,
$\eta_k(s) := \sum_{i=k}^{n} \frac{a_i}{(s+1)^{i+1-k}} I$, with the a_i's given by (2.62)

Optimal Controller Design with Closed-Loop Parameterizations

It follows directly from the parameterization provided in Theorem 2.5.4 that the closed-loop mappings Φ^x, Φ^u given by (2.70) will have the same band size as the parameter Θ. This allows the structured $\mathcal{H}_2$ design problem (2.15) to be written in terms of Θ as:

$$\begin{aligned} &\left\{ \begin{array}{ll} \inf\limits_{\theta \in \overline{\mathcal{R}}_s} & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} F\Theta + L \\ \chi\Theta + \eta \end{bmatrix} B_1 e_j \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & \Theta \text{ spatially-invariant with band size } M \end{array} \right. \\ = &\left\{ \begin{array}{ll} \inf\limits_{\Theta \in \overline{\mathcal{R}}_s} & \left\| \left(C_1 L + D_{12}\eta\right) B_1 e_j + \left(C_1 F + D_{12}\chi\right) \ \Theta \ B_1 e_j \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & \Theta \text{ spatially-invariant with band size } M \end{array} \right. \end{aligned} \tag{2.72}$$

where F, L and η are defined in Theorems 2.5.3 and 2.5.4. Assuming that C_1 and D_{12} have finite band size, $(C_1 L + D_{12}\eta)\, B_1 e_j$ has only finitely many nonzero entries. Define H to be the vector composed of these nonzero entries of $(C_1 L + D_{12}\eta)\, B_1 e_j$. Then (2.72) can be written in the form of a standard, unconstrained model matching problem:

$$\inf_{Q \in \overline{\mathcal{R}}_s} \|H + UQV\|_{\mathcal{H}_2}^2, \tag{2.73}$$

with $(2M+1)$ transfer function parameters $Q := \begin{bmatrix} \theta_{-M}(s) & \cdots & \theta_0(s) & \cdots & \theta_M(s) \end{bmatrix}$, and U, V constructed accordingly from $(C_1 F + D_{12}\chi)$ and $B_1 e_j$.

Thus, even in the infinite space setting, (2.15) can be written as a standard *finite-dimensional* model matching problem, when finite band size constraints are imposed on the closed-loop mappings. The number of transfer function parameters in this problem is $(2M+1)$ where M is the constrained closed-loop band size. In Sections 2.5.4 and 2.5.6, we will provide examples of this procedure.

2.5.4 Consensus of First-Order Systems via System Level Parameterizations

In this section we apply the results of Section 2.5.2 to provide analytic solutions to the consensus problem (2.53). We consider a performance output of the form

$$z = \begin{bmatrix} y \\ \gamma u \end{bmatrix} = \begin{bmatrix} Cx \\ \gamma u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} w,$$

with C spatially-invariant, and solve the $\mathcal{H}_2$ design problem subject to closed-loop spatial sparsity constraints:

$$\left.\begin{array}{ll} \inf\limits_{K \text{ stabilizing}} & \left\| \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \Phi^x \\ \Phi^u \end{bmatrix} e_j \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & K \text{ spatially-invariant} \\ & \Phi^x, \Phi^u \text{ have band size } M \\ & \text{System Dynamics (2.53),} \end{array}\right\} \tag{2.74}$$

where we optimize the $\mathcal{H}_2$ norm of any spatial site j, due to spatial invariance (see Equations (2.9), (2.10)). In the finite space setting ($\mathbb{G} = \mathbb{Z}_N$), we restrict $M < \frac{N}{2}$ so the band size constraint is nontrivial, and in the infinite space setting allow for any finite choice of M.

Let K be a spatially-invariant system. Then, by Theorem 2.5.1, the controller $u = Kx$ is stabilizing for the spatially-invariant locally 1st order system (2.53) if and only if the corresponding closed-loop mappings are of the form

$$\begin{aligned} \Phi^x &= (sI + I)^{-1}\Theta + (sI + I)^{-1} \\ \Phi^u &= s(sI + I)^{-1}\Theta - (sI + I)^{-1}, \end{aligned} \tag{2.75}$$

for some spatially-invariant system $\Theta = \{\theta_n\}_{n \in \mathbb{G}} \in \overline{\mathcal{R}}_s$. Optimization problem (2.74) to be written in terms of Θ as:

$$J^{\text{opt}} := \left\{ \begin{array}{ll} \inf_{\theta \in \overline{\mathcal{R}}_s} & J = \left\| \begin{bmatrix} \frac{1}{s+1} C e_j \\ \frac{-\gamma}{s+1} e_j \end{bmatrix} + \begin{bmatrix} \frac{1}{s+1} C \\ \frac{\gamma s}{s+1} I \end{bmatrix} \Theta e_j \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & \Theta \text{ spatially-invariant,} \\ & \Theta \text{ has band size } M. \end{array} \right. \tag{2.76}$$

Remark 2 *A problem of interest for consensus applications is to restrict the controller to only have access to only* relative *state measurements, e.g. measurements of the form* $x_i - x_j$*. However, in Chapter 3 it will be demonstrated that when finite band size constraints are imposed on the closed loops, there does not exist a relative feedback controller that achieves a finite solution to* (2.74)*. Thus the problem of relative feedback controller design is not addressed in this chapter.*

We next demonstrate that (2.76) can be converted to a standard unconstrained finite-dimensional model matching problem with $(2M + 1)$ transfer function parameters, as shown in Section 2.5.3.

As an illustrative example, we consider the specific case $M = 1$. In this case, (2.76) can be written as

$$J^{\text{opt}} = \begin{cases} \inf\limits_{\theta_{-1},\theta_0,\theta_1} & \left\| \begin{bmatrix} \frac{1}{s+1}Ce_1 \\ \frac{-\gamma}{s+1}e_1 \end{bmatrix} + \begin{bmatrix} \frac{1}{s+1}C \\ \frac{\gamma s}{s+1}I \end{bmatrix} \begin{bmatrix} \theta_0 \\ \theta_1 \\ 0 \\ \vdots \\ 0 \\ \theta_{-1} \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & \theta_{-1},\theta_0,\theta_1 \in \overline{\mathcal{R}}_s \end{cases} \tag{2.77}$$

$$= \begin{cases} \inf\limits_{\theta_{-1},\theta_0,\theta_1} & \left\| \begin{bmatrix} \frac{1}{s+1}Ce_1 \\ \frac{-\gamma}{s+1}e_1 \end{bmatrix} + \tilde{U} \begin{bmatrix} \theta_{-1} \\ \theta_1 \\ \theta_0 \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & \theta_{-1},\theta_0,\theta_1 \in \overline{\mathcal{R}}_s \end{cases}$$

where $\tilde{U} := \begin{bmatrix} \tilde{u}_{-1} & \tilde{u}_0 & \tilde{u}_1 \end{bmatrix}$ with $\tilde{u}_{-1}, \tilde{u}_0, \tilde{u}_1$ the last, second and first column of $\begin{bmatrix} \frac{1}{s+1}C \\ \frac{\gamma s}{s+1}I \end{bmatrix}$, respectively. We write (2.77) as a standard model-matching problem:

$$J^{\text{opt}} = \inf_{Q \in \overline{\mathcal{R}}_s} \|H + UQ\|_{\mathcal{H}_2}^2 \,, \tag{2.78}$$

where H and U are formed by the nonzero rows of $\begin{bmatrix} \frac{1}{s+1}Ce_1 \\ \frac{-\gamma}{s+1}e_1 \end{bmatrix}$ and $\tilde{U}$, respectively. A similar formulation holds for more general $M \neq 1$.

We next provide examples that demonstrate that the optimal solution of (2.78) can be computed analytically using the techniques of [31], when an inner-outer factorization $U = U_i U_o$ is available.

Analytic Solutions

We consider two measures of consensus - *local error* and *deviation from average*, defined as follows:

- **Local Error**

  $$y_n := x_n - x_{n-1}.$$

  Compactly, we write $y = C^{\text{LE}}x$, where C^{LE} is the spatial convolution operator with convolution kernel c defined by $c_0 = 1$, $c_1 = -1$ and $c_k = 0$ for all $k \notin \{0, 1\}$. C^{LE} can be represented in the finite space setting by the circulant matrix

  $$C^{\text{LE}} := \begin{bmatrix} 1 & 0 & \cdots & 0 & -1 \\ -1 & 1 & \cdots & 0 & 0 \\ & \ddots & \ddots & & \\ 0 & 0 & \cdots & -1 & 1 \end{bmatrix},$$

- **Deviation from Average**

$$y_n := x_n - \frac{1}{N}\sum_{m=0}^{N-1} x_m.$$

Compactly we write $y = C^{\text{Ave}}x$. In the finite space setting, C^{Ave} is given by the circulant matrix

$$C^{\text{Ave}} := \frac{1}{N}\cdot\begin{bmatrix} N-1 & -1 & \cdots & -1 \\ -1 & N-1 & \cdots & -1 \\ & & \ddots & \\ -1 & -1 & \cdots & N-1 \end{bmatrix}.$$

The case of a local error measure with closed-loop band size one constraint is analyzed in the following lemma and theorem.

Lemma 2.5.5 *The optimal $\Theta = \{\theta_n(s)\} \in \overline{\mathcal{R}}_s$ which solves (2.76) for $C = C^{LE}$ with a band size $M = 1$ constraint is given by:*

$$\begin{aligned}
\theta_0 &= \frac{-1}{2}\left(\frac{-\alpha}{(2-\sqrt{2})^{1/2}+\gamma s} + \frac{\beta}{(2+\sqrt{2})^{1/2}+\gamma s}\right),\\
\theta_1 &= \frac{1}{2\sqrt{2}}\left(\frac{-\alpha}{(2-\sqrt{2})^{1/2}+\gamma s} + \frac{\beta}{(2+\sqrt{2})^{1/2}+\gamma s}\right),\\
\theta_{-1} &= \theta_1
\end{aligned}$$

where $\alpha := (2-\sqrt{2})^{1/2} - \gamma$, $\beta := (2+\sqrt{2})^{1/2} - \gamma$. Furthermore, the corresponding optimal closed-loop norm per spatial site is given by

$$J = \frac{\gamma}{4}\left((2-\sqrt{2})^{1/2} + (2+\sqrt{2})^{1/2}\right). \tag{2.79}$$

A proof of Lemma 2.5.5 is provided in Appendix 2.7.3.

Theorem 2.5.6 *The optimal closed-loops $\Phi^x = \{\phi_n^x\}_{n\in\mathbb{G}}$, $\Phi^u = \{\phi_n^u\}_{n\in\mathbb{G}}$ corresponding to (2.76) for $C = C^{LE}$ with a band size constraint $M = 1$ are given by:*

$$\begin{aligned}
\phi_0^x &= \frac{1}{g(s)}\left(\left(2-\sqrt{2}\right)^{1/2} + \left(2+\sqrt{2}\right)^{1/2} + 2\gamma s\right)\\
\phi_1^x = \ \phi_{-1}^x &= \frac{1}{\sqrt{2}g(s)}\left(\left(2+\sqrt{2}\right)^{1/2} - \left(2-\sqrt{2}\right)^{1/2}\right)\\
\phi_0^u &= \frac{1}{g(s)}\left(-2\sqrt{2} - \gamma s\left(\left(2-\sqrt{2}\right)^{1/2} + \left(2+\sqrt{2}\right)^{1/2}\right)\right)\\
\phi_1^u = \ \phi_{-1}^u &= \frac{s}{\sqrt{2}g(s)}\left(\left(2+\sqrt{2}\right)^{1/2} - \left(2-\sqrt{2}\right)^{1/2}\right)
\end{aligned} \tag{2.80}$$

where $g(s) := 2\left(\left(2-\sqrt{2}\right)^{1/2} + \gamma s\right)\left(\left(2+\sqrt{2}\right)^{1/2} + \gamma s\right)$.

Proof: Analytic expressions for the optimal closed-loop maps are determined by the parameters $\theta_{-1},\ \theta_0,\ \theta_1$ using equation (2.75) as

$$\begin{bmatrix} \phi^x_{-1}(s) \\ \phi^x_0(s) \\ \phi^x_1(s) \end{bmatrix} = \frac{1}{s+1}\left(\begin{bmatrix} \theta_{-1}(s) \\ \theta_0(s) \\ \theta_1(s) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)$$
$$\begin{bmatrix} \phi^u_{-1}(s) \\ \phi^u_0(s) \\ \phi^u_1(s) \end{bmatrix} = \frac{1}{s+1}\left(s\begin{bmatrix} \theta_{-1}(s) \\ \theta_0(s) \\ \theta_1(s) \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}\right), \tag{2.81}$$

■

Similar results hold for more general choice of $M \neq 1$.

Analytic solutions for the optimal closed-loop norm (2.79) for this case, as well as for the deviation from average metric ($C = C^{\text{Ave}}$) and band size constraint $M = 2$ are summarized in Table 2.1. Analytic expressions for the optimal parameter θ (as in Lemma 2.5.5) corresponding to other entries of Table 2.1 are provided in Appendix 2.7.4.

Table 2.1

	Optimal Closed-Loop Norm (per spatial cite)
Local Error	
Sparsity $M = 1$	$\frac{\gamma}{4}\left((2-\sqrt{2})^{1/2} + (2+\sqrt{2})^{1/2}\right)$ $\approx 0.653 \cdot \gamma$
Sparsity $M = 2$	$\frac{\gamma}{6}\left((2-\sqrt{3})^{1/2} + \sqrt{2} + (2+\sqrt{3})^{1/2}\right)$ $\approx 0.644 \cdot \gamma$
Deviation From Ave.	
Sparsity $M = 1$	$\gamma\left(\frac{1}{3} + \frac{1}{6}\sqrt{1-\frac{3}{N}}\right)$
Sparsity $M = 2$	$\gamma\left(\frac{1}{8} + \frac{1}{10}\sqrt{1-\frac{5}{N}}\right)$

Note that C^{LE} has band size one and C^{Ave} does not have finite band size. Due to the added stochastic noise $\{w_n\}$, the scaling in network size of the solution to (2.74) will differ based on choice of $C = C^{\text{LE}}$ or $C = C^{\text{Ave}}$, as demonstrated in Table 2.1). The optimal closed-loop norm per spatial site for the local error metric is independent of the number of subsystems N and holds even for the infinite space setting. In contrast, the optimal closed-loop norm for the deviation from average metric increases with the number of subsystems N. We conjecture that the limit of this expression as $N \to \infty$ provides the closed-loop norm per spatial site for the infinite space setting, but we do not formalize this in this chapter.

2.5.5 Structured Controller Implementation

The closed-loop band size of M constraint in (2.15) ensures that a disturbance entering into any subsystem does not affect neighboring subsystems more than a distance of M away *for all time* - e.g. a band size of one means that disturbances at subsystem n may only affect subsystems $(n-1), n, (n+1)$ for all time. This disturbance *localization* property is useful in certain applications, e.g. control of the power grid, but in many applications the true benefit of this constraint is that this closed-loop band size carries over to an *implementation* of the resulting controller.

We illustrate this by analyzing the controller implementation for the consensus problem with a local error measure and a band size one constraint. We begin by introducing the following definition to formally analyze one notion of structure of a *realization* or *implementation* of a system. Such notions of structured implementations will be further studied in Chapter 3.

Definition 2.5.1 *A realization $K(s) = C(sI - A)^{-1}B + D$ of the finite-dimensional transfer matrix K is said to be* structured with band size M *if the matrices A, B, C, D each have band size M. Similarly, a spatially-invariant system K with input x and output u is said to have a* structured implementation with band size M *if the output at spatial location n, $x_n(t)$, can be computed using only inputs $u_m(\tau)$ with $|m-n| < M$ and $\tau \leq t$.*

We remark that requirements of stabilizability and detectability are not included in definition 2.5.1, and structured implementations will likely be non-minimal.

Lemma 2.5.7 *Let Φ^x and Φ^u be the optimal closed-loop mappings given by* (2.80)*. The following dynamics define a structured implementation of the corresponding controller $K = \Phi^u(\Phi^x)^{-1}$ with band size one:*

$$\begin{aligned} \begin{bmatrix} \dot{\xi}_m \\ \dot{\zeta}_m \end{bmatrix} &= \begin{bmatrix} A_x + B_{x,0}C_x & 0 \\ F_0 C_x & A_x \end{bmatrix} \begin{bmatrix} \xi_m \\ \zeta_m \end{bmatrix} + \begin{bmatrix} B_{x,0} \\ F_0 \end{bmatrix} x_m + \begin{bmatrix} B_{x,1} \\ F_1 \end{bmatrix} C_x(\xi_{m-1} + \xi_{m+1}) \\ &\quad + \begin{bmatrix} B_{x,1} \\ F_1 \end{bmatrix} (x_{m-1} + x_{m+1}) \\ u_n &= \begin{bmatrix} C_x F_0 C_x & C_x A_x \end{bmatrix} \begin{bmatrix} \xi_m \\ \zeta_m \end{bmatrix} + C_x F_0 x_m + C_x F_1 C_x(\xi_{m-1} + \xi_{m+1}) \\ &\quad + C_x F_1 (x_{m-1} + x_{m+1}), \end{aligned} \tag{2.82}$$

where $\vartheta_m := \begin{bmatrix} \xi_m^T & \zeta_m^T \end{bmatrix}^T$ is the local state of the subcontroller at spatial cite m, and $A_x, B_{x,i}, C_x$, and F_i are finite dimensional matrices which will be defined in (2.86)*.*

With implementation (2.82), the state of subcontroller m is computed using only subcontroller states and plant subsystem states from nearest neighbors, i.e. $\vartheta_j \begin{bmatrix} \xi_j^T & \zeta_j^T \end{bmatrix}^T$ and x_j for $j = m, (m-1), (m+1)$. (see Figure 2.2). Note that the controller itself, $K = \Phi^u(\Phi^x)^{-1}$, in general does *not* have finite band size, according to Definition 2.2.2.

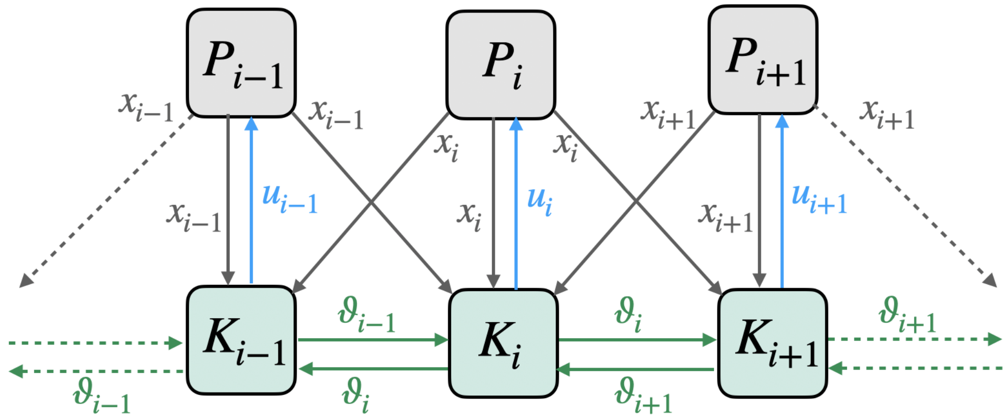


Figure 2.2: Implementation of optimal controller with spatial sparsity extent of one imposed on the closed-loop mappings. Each subplant P_i sends its local state x_i to subcontrollers K_i, K_{i-1}, and K_{i+1}. Each subcontroller K_i sends its local state $\vartheta_i := \begin{bmatrix}\xi_i^T & \zeta_i^T\end{bmatrix}$ to neighboring subcontrollers K_{i-1} and K_{i+1}. Each subcontroller K_i provides the local control action u_i to plant P_i based on this information exchange.

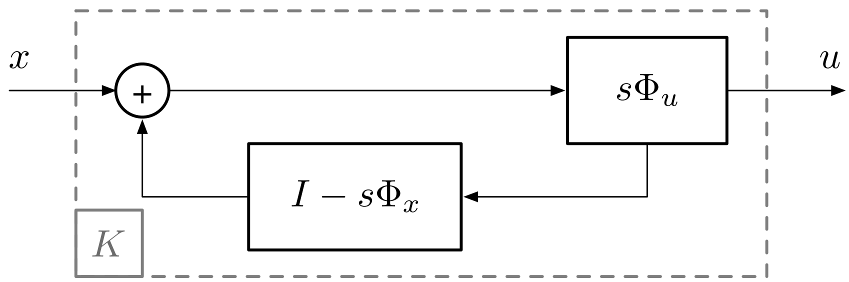


Figure 2.3: Implementation of controller $K = \Phi^u(\Phi^x)^{-1}$ via closed-loop mappings to preserve structure.

This highlights the difference between the structure of a controller *transfer function* and the structure of a corresponding *realization*; this distinction has been recently emphasized in e.g. [12–14], and will be further considered in Chapter 3.

Proof: We leverage the implementation suggested in [11], and implement the controller $u = Kx$ as

$$\begin{aligned} v &= (I - s\Phi^x)v + x && := \tilde{\Phi}^x v + x \\ u &= (s\Phi^u)x && := \tilde{\Phi}^u x, \end{aligned} \tag{2.83}$$

(see Figure 2.3). The spatially-invariant system $\tilde{\Phi}^x := (I - s\Phi^x)$ has the same band size as Φ^x, and thus is defined by its three nonzero components:

$$\begin{aligned} \tilde{\phi}^x_{-1} &= \frac{1}{g(s)}\left(2\sqrt{2} + (a+b)\,\gamma s\right) \\ \tilde{\phi}^x_0 &= \frac{\gamma}{\sqrt{2}g(s)}\,(a-b) \\ \tilde{\phi}^x_1 &= \frac{1}{g(s)}\left(2\sqrt{2} + (a+b)\,\gamma s\right), \end{aligned} \tag{2.84}$$

where $a := (2-\sqrt{2})^{1/2},\ b := (2+\sqrt{2})^{1/2}$. Similarly, $\tilde{\Phi}^u$ is defined in terms of its three nonzero components:

$$\begin{aligned}\tilde{\phi}^u_{-1} &= -s^2\tilde{\phi}^x_1,\\ \tilde{\phi}^u_0 &= -s\tilde{\phi}^x_0,\\ \tilde{\phi}^u_1 &= -s^2\tilde{\phi}^x_1.\end{aligned} \tag{2.85}$$

Note that $\tilde{\Phi}^x, \tilde{\Phi}^u \in \overline{\mathcal{R}}_s$. Let $\tilde{x}$ denote the output of $\tilde{\Phi}^x$, i.e. $\tilde{x} = \tilde{\Phi}^x v$. Then, the n^{th} component of this output, $\tilde{x}_n$ and the output of the n^{th} subcontroller u_n are given by

$$\begin{bmatrix} \tilde{x}_n \\ u_n \end{bmatrix} = \begin{bmatrix} \tilde{\phi}^x_{-1} & \tilde{\phi}^x_0 & \tilde{\phi}^x_1 \\ \tilde{\phi}^u_{-1} & \tilde{\phi}^u_0 & \tilde{\phi}^u_1 \end{bmatrix} \begin{bmatrix} v_{n-1} \\ v_n \\ v_{n+1} \end{bmatrix}.$$

The following are realizations of $\begin{bmatrix} \tilde{\phi}^x_{-1} & \tilde{\phi}^x_0 & \tilde{\phi}^x_1 \end{bmatrix}$ and $\begin{bmatrix} \tilde{\phi}^u_{-1} & \tilde{\phi}^u_0 & \tilde{\phi}^u_1 \end{bmatrix}$:

$$\begin{aligned}\begin{bmatrix} \tilde{\phi}^x_{-1} & \tilde{\phi}^x_0 & \tilde{\phi}^x_1 \end{bmatrix} &= \left[\begin{array}{c|ccc} A_x & B_{x,1} & B_{x,0} & B_{x,1} \\ \hline C_x & 0 & 0 & 0 \end{array}\right]\\ \begin{bmatrix} \tilde{\phi}^u_{-1} & \tilde{\phi}^u_0 & \tilde{\phi}^u_1 \end{bmatrix} &= \left[\begin{array}{c|ccc} A_x & F_1 & F_0 & F_1 \\ \hline C_xA_x & C_xF_1 & C_xF_0 & C_xF_1 \end{array}\right],\end{aligned}$$

where

$$\begin{aligned} A_x &:= \begin{bmatrix} 0 & 1 \\ \frac{-ab}{\gamma^2} & \frac{-(a+b)}{\gamma} \end{bmatrix}, \quad B_{x,1} = \begin{bmatrix} 0 \\ \frac{\gamma(a-b)}{\sqrt{2}} \end{bmatrix},\\ B_{x,0} &= \begin{bmatrix} \frac{a+b}{2\gamma} \\ 2\sqrt{2} - \frac{(a+b)^2}{2\gamma} \end{bmatrix}, \quad C_x := \begin{bmatrix} 1 & 0 \end{bmatrix},\\ F_0 &:= \begin{bmatrix} -\frac{a+b}{2\gamma} \\ (a+b)^2 - \frac{\sqrt{2}}{\gamma^2} \end{bmatrix}, \quad F_1 := \begin{bmatrix} \frac{a-b}{s\sqrt{2}\gamma^2} \\ \frac{-(a-b)(a+b)}{2\sqrt{2}\gamma^2} \end{bmatrix}.\end{aligned} \tag{2.86}$$

We write out the local dynamics of the controller K at each spatial cite using these realizations to obtain (2.82).

■

We note that the realization (2.82) is not relative, i.e. to implement the controller in this way, *absolute* measurements of subsystem state and control actions are required. Thus, this implementation does not provide a solution to the *relative* feedback control problem, which we address in Chapter 3.

2.5.6 Control of Vehicular Formations using Backstepping Parameterizations

We apply the parameterization presented in Theorem 2.5.4 to the vehicle formation problem. We write the dynamics (2.54) compactly as:

$$\dot{x} = Ax + B_1 w + B_2 u, \tag{2.87}$$

where A, B_1, B_2 are pointwise multiplication operators defined by the matrices $a := \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$, $b_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $b_2 := \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Note that (2.87) is a spatially-invariant locally 2^{nd} order system with $(A + I, B_2)$ in controllable-canonical form. By Theorem 2.5.4, a spatially-invariant controller K is stabilizing for (2.87) if and only if the corresponding closed-loop mappings $\Phi^x = \{\phi_n^x\}_{n\in\mathbb{G}}$ and $\Phi^u = \{\phi_n^u\}_{n\in\mathbb{G}}$ are of the form:

$$\begin{aligned}
\Phi_n^x(s) &= \begin{cases} F(s)\begin{bmatrix} \theta_{1,n}(s) & \theta_{2,n}(s) \end{bmatrix} + L(s), & n = 0 \\ F(s)\begin{bmatrix} \theta_{1,n}(s) & \theta_{2,n}(s) \end{bmatrix}, & n \neq 0 \end{cases} \\
\Phi_n^u(s) &= \begin{cases} \chi(s)\begin{bmatrix} \theta_{1,n}(s) & \theta_{2,n}(s) \end{bmatrix} + \begin{bmatrix} \eta_1(s) & \eta_2(s) \end{bmatrix}, & n = 0 \\ \chi(s)\begin{bmatrix} \theta_{1,n}(s) & \theta_{2,n}(s) \end{bmatrix}, & n \neq 0 \end{cases}
\end{aligned} \tag{2.88}$$

with $F(s) = \begin{bmatrix} \frac{1}{s+1} \\ \frac{1}{(s+1)^2} \end{bmatrix}$, $L(s) = \begin{bmatrix} \frac{1}{s+1} & 0 \\ \frac{1}{(s+1)^2} & \frac{1}{s+1} \end{bmatrix}$, $\chi(s) = \frac{s^2}{(s+1)^2}$, and $\eta_1(s) = \frac{-2s-1}{(s+1)^2}$.

With a performance output z of the form $z = \begin{bmatrix} y^T & \gamma u^T \end{bmatrix}^T$ with $y = Cx$, the optimal $\mathcal{H}_2$ design problem subject to band size constraints (2.15) can be written using (2.88):

$$\begin{aligned}
& \inf_{\theta_1,\theta_2 \in \overline{\mathcal{R}}_s} \left\| \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} F\begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} + L \\ \chi\begin{bmatrix} \theta_1 & \theta_2 \end{bmatrix} + \eta \end{bmatrix} B_1 \right\|_{\mathcal{H}_2}^2 \\
& \text{s.t.} \quad \theta_1, \theta_2 \text{ spatially-invariant with band size } M \\
= & \inf_{\theta_1 \in \overline{\mathcal{R}}_s} \left\| \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} F\theta_1 + F \\ \chi\theta_1 + \eta_1 \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \\
& \text{s.t.} \quad \theta_1 \text{ spatially-invariant with band size } M \\
= & \inf_{\theta_1 \in \overline{\mathcal{R}}_s} \left\| \begin{bmatrix} CF \\ \gamma\eta_1 \end{bmatrix} + \begin{bmatrix} CF \\ \gamma\chi I \end{bmatrix} \theta_1 \right\|_{\mathcal{H}2}^2 \\
& \text{s.t.} \quad \theta_1 \text{ spatially-invariant with band size } M
\end{aligned} \tag{2.89}$$

where F, L, η_1 represent pointwise multiplication by the finite dimensional transfer matrices $F(s), L(s), \eta_1(s)$. We consider C corresponding to one of the following measures of consensus:

- **Local error of vehicle position:**

$$y_n = \begin{bmatrix} 0 & 1 \end{bmatrix} (x_n - x_{n-1}).$$

  Compactly, $y = C^{\text{LE}}x$ where C^{LE} is the spatial convolution operator with convolution kernel $c_0 = \begin{bmatrix} 1 & 0 \end{bmatrix}, c_1 = \begin{bmatrix} -1 & 0 \end{bmatrix}, c_n = 0$ for $n \neq 0, 1$.

- **Deviation from average of vehicle position:**

$$y_n = \begin{bmatrix} 0 & 1 \end{bmatrix} x_n - \frac{1}{N}\sum_{i=1}^{N} \begin{bmatrix} 0 & 1 \end{bmatrix} x_i,$$

with N the number of subsystems. Compactly, $y = C^{\text{Ave}}x$ where C^{Ave} is the spatial convolution operator with convolution kernel $c_0 = \left(1 - \frac{1}{N}\right)\begin{bmatrix} 1 & 0 \end{bmatrix}, c_n = -\frac{1}{N}\begin{bmatrix} 1 & 0 \end{bmatrix}$ for $n \neq 0$.

For either of these choices of C, following the procedure of Section 2.5.3, (2.89) can be reduced to an unconstrained model-matching problem of the form:

$$\inf_{Q \in \overline{\mathcal{R}}_s} \quad \|H + UQ\|_{\mathcal{H}_2}^2 , \tag{2.90}$$

with $(2M+1)$ transfer function parameters $Q := \begin{bmatrix} \theta_{1,-M} & \cdots & \theta_{1,M} \end{bmatrix}^T$, and H and U constructed from $\begin{bmatrix} CF \\ \gamma\eta_1 \end{bmatrix}$ and $\begin{bmatrix} CF \\ \gamma\chi I \end{bmatrix}$ respectively. We solve (2.90) numerically for various choices of band size constraint M.

The results for the case of a control weight $\gamma = 3$ and $N = 121$ subsystems are illustrated in Figure 2.4. The optimal closed-loop $\mathcal{H}_2$ cost is plotted as a function of closed-loop band size M for a local error objective (top) and a deviation from average objective (bottom). We note that as band size M increases, the closed-loop maps have less constrained structure, and the corresponding closed-loop cost decreases toward the unconstrained optimal illustrated by the red lines. In the local error case (top), the convergence appears exponential, as further demonstrated by a logarithmic plot which appears roughly linear (see Figure 2.5). In contrast, the convergence rate for the deviation from average measure (bottom) is not exponential. Quantifying this decay rate and understanding the differences in decay rate for these two measures is the subject of future work. It is known that the optimal unconstrained controller for the local error measure is defined by a static spatially-invariant matrix with entries that decay exponentially off the diagonal [16]. If the corresponding closed-loops for the unconstrained problem have a similar decay property, this could explain the exponential convergence rate observed. It remains an open question how to formally define this decay rate for dynamic closed-loop mappings and prove whether the closed-loops satisfy this property.

2.5.7 Closed-Loop Parameterizations for Spatially-Varying Systems

In this section, we present an explicit parameterization of all achievable stabilized closed-loop mappings for *spatially-varying systems*, generalizing the results of Sections 2.5.2 and 2.5.3. We begin by analyzing a system composed of first-order subsystems with decoupled dynamics.

Locally 1^{st} Order Systems

First, we consider distributed systems of the form

$$\dot{x}_n(t) = a_n x_n(t) + w_n(t) + b_n u_n(t), \ n \in \mathbb{Z}_N, \tag{2.91}$$

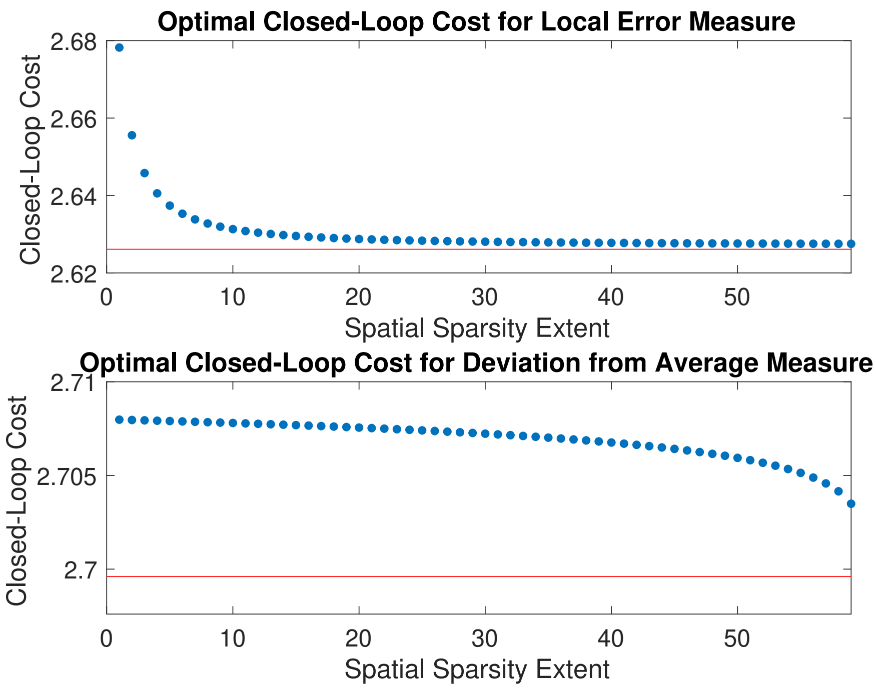


Figure 2.4: The optimal closed-loop cost of structured optimal controller design problem (2.89) for the vehicle consensus problem is plotted against the spatial sparsity extent constraint M imposed on the closed-loop mappings for a local error metric (top) and deviation from average metric (bottom), for $N = 121$ subsystems and a control cost weighting of $\gamma = 3$. The red lines illustrate the optimal closed-loop cost with no spatial sparsity constraints imposed.

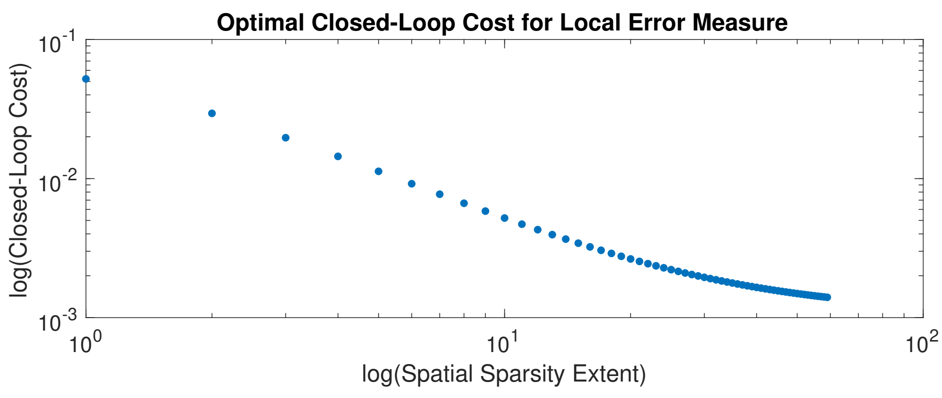


Figure 2.5: The optimal closed-loop cost of structured optimal controller design problem (2.89) for the vehicle consensus problem with a local error performance metric is plotted against the spatial sparsity extent constraint M imposed on the closed-loop mappings on a log log scale.

where x_n, w_n, and u_n denote the scalar-valued state, exogenous disturbance, and control signal at the n^{th} site, respectively. We assume that (2.91) is controllable so that $b_n \neq 0$ for each n. We call this class of systems *locally 1^st order*. This is a generalization of the class of spatially-invariant locally 1st order systems presented in Section 2.5.4.

At each spatial site, the local state, x_n, is scalar-valued, but the overall state x is a vector of dimension N composed of all the local states. In vector form, (2.91) is written as:

$$\dot{x}(t) \;=\; Ax(t) \;+\; w(t) \;+\; B_2 u(t),$$

the "A-matrix" and "B-matrix", $A := \text{diag}\{a_n\}_{n\in\mathbb{Z}_N}$ and $B_2 := \text{diag}\{b_n\}_{n\in\mathbb{Z}_N}$, are finite-dimensional diagonal matrices. For an "$A-$matrix" of this form,

$$(sI - A)^{-1} = (sI - \text{diag}\{a_n\})^{-1} = \text{diag}\left\{\frac{1}{s-a_n}\right\}, \tag{2.92}$$

so that the relation (2.59) is simple to state in terms of the rows of the relevant matrices as

$$\text{row}_n(\Phi^x) = \frac{1}{s-a_n}\ \text{row}_n(I + B_2\Phi^u) \tag{2.93}$$

Using (2.93), we derive an explicit parameterization of all achievable stabilized closed-loop mappings for locally 1st order systems, as stated in the following theorem.

Theorem 2.5.8 *A (dynamic or static) controller $u = Kx$ is stabilizing for* (2.91) *if and only if the resulting closed-loop mappings are of the form*

$$\Phi^x(s) \;=\; B\cdot\text{diag}\{\gamma_n\}\Theta(s) + \text{diag}\{\gamma_n\} \tag{2.94}$$

$$\Phi^u(s) \;=\; \text{diag}\{\alpha_n\}\Theta(s) + \text{diag}\{\beta_n\}, \tag{2.95}$$

for some $\Theta \in \overline{\mathcal{R}}_s$, where $B := \text{diag}\{b_n\}$, and α_n, γ_n, and β_n are defined as follows:

$$\begin{cases} \alpha_n := 1,\ \beta_n := 0,\ \gamma_n := \frac{1}{s-a_n}, & \textit{if } \Re(a_n) < 0, \\ \alpha_n := \frac{s-a_n}{b_n(s+1)},\ \beta_n := -\frac{a_n+1}{b_n(s+1)},\ \gamma_n := \frac{1}{s+1}, & \textit{else.} \end{cases}$$

A proof of this result is provided in Appendix 2.7.5.

We next extend the result of Theorem 2.5.3 to the case of higher order subsystems, generalizing the results of Section 2.5.3 to the spatially-varying setting.

Locally Finite Dimensional Subsystems

In this section we consider distributed systems of the form:

$$\dot{x}_m = A^{(m)}x_m + B_1^{(m)}w_m + B_2^{(m)}u_m,\ m \in \mathbb{Z}_N, \tag{2.96}$$

with each $(A^{(m)} + I, B_2^{(m)})$ in controllable canonical form. We let n_m denote the order of subsystem m, so that $x_m = \begin{bmatrix} x_{m1}^T & x_{m2}^T & \cdots & x_{mn_m}^T \end{bmatrix}^T$ for each m, and refer to systems of the form (2.96) as *locally finite-dimensional.*

In vector form (2.96) can be written as

$$\dot{x} = Ax + B_1 w + B_2 u,$$

where $A = \mathrm{diag}\{A^{(m)}\}_{m\in\mathbb{Z}_N}$, $B_2 = \mathrm{diag}\{B_2^{(m)}\}_{m\in\mathbb{Z}_N}$, with each $A^{(m)}$ and $B^{(m)}$ of the form

$$(A^{(m)} + I) = \begin{bmatrix} -a_1^{(m)} I & -a_2^{(m)} I & \cdots & -a_{n_m}^{(m)} I \\ I & 0 & \cdots & 0 \\ & \ddots & \ddots & \\ & & & 0 \end{bmatrix},$$

$$B_2^{(m)} = \begin{bmatrix} I & 0 & 0 & \cdots & 0 \end{bmatrix}^T$$

where n_m is the order of subsystem m.

Theorem 2.5.9 *A (dynamic or static) controller $u = Kx$ is stabilizing for the locally finite-dimensional system* (2.96) *if and only if the resulting closed-loop mappings are given by*

$$\phi_{ij}^x(s) = \begin{cases} F(s)\theta_{ij}(s) + L(s), & i = j \\ F(s)\theta_{ij}(s), & i \neq j \end{cases}$$
$$\phi_{ij}^u(s) = \begin{cases} \chi^i(s)\theta_{ij}(s) + \eta^i(s), & i = j \\ \chi^i(s)\theta_{ij}(s), & i \neq j \end{cases} \tag{2.97}$$

where ϕ_{ij}^x and ϕ_{ij}^u are the (i,j) entries of Φ^x and Φ^u repectively, $F(s)$ and $L(s)$ are defined in (2.63), *$\chi_i(s) := 1 + \frac{a_1^i}{s+1} + \cdots + \frac{a_{n_i}^i}{(s+1)^{n_i}}$, and $\eta^i := \begin{bmatrix} \eta_1^i(s) & \cdots & \eta_{n_i}^i(s) \end{bmatrix}$ with $\eta_k^i := \sum_{\ell=k}^{n_i} \frac{a_\ell^i}{(s+1)^{\ell+1-k}} I$.*

A proof of this result is in Appendix 2.7.6.

2.6 Conclusion

In this chapter, we derived *explicit* parameterizations of all achievable stabilized closed-loop mappings for certain subclasses of finite-dimensional and (finite or infinite extent) spatially-invariant systems. In Part 1, we analyzed systems which are stable in open-loop or have an invertible control to state operator. In Part 2, we analyzed distributed systems with decoupled open-loop dynamics which are controllable.In contrast to the *implicit* parameterization introduced by SLS, our explicit parameterizations eliminate the need for temporal FIR approximations, allowing the $\mathcal{H}_2$ design problem to be converted to a standard unconstrained model matching problem which may lead to analytic IIR solutions. We studied the consensus of first-order subsystems and the vehicular formation problem. These examples allowed us to comment on performance scalings with system size and structural constraints, and analyze resulting controller implementations. Future work includes extending the parameterizations provided in this paper to distributed systems with *coupled* subsystem dynamics and formally analyzing convergence rates observed numerically.

2.7 Appendix

2.7.1 Proof of Theorem 2.5.1:

First assume that K is stabilizing for (2.55) in the infinite space setting $\mathbb{G} = \mathbb{Z}$. Then following [20], the definitions (2.13) of $\Phi^x = \{\phi_n^x(s)\}_{n\in\mathbb{Z}}$ and $\Phi^u = \{\phi_n^u(s)\}_{n\in\mathbb{Z}}$ show that

$$(sI - aI)\Phi^x - \Phi^u = I. \tag{2.98}$$

The relation (2.98) can be written in terms of the convolution kernels of the closed-loop mappings as,

$$(s-a)\phi_n^x(s) - \phi_n^u(s) = 1, \quad \text{for all } n \in \mathbb{Z}.$$

Then by the arguments presented for the finite space setting, ϕ_n^x, and ϕ_n^u are of the form:

$$\begin{aligned} \phi_n^x(s) &= \frac{1}{s+1}\theta_n(s) - \frac{1}{s+1}I, \\ \phi_n^u(s) &= \frac{s-a}{s+1}\theta_n(s) - \frac{a+1}{s+1}I. \end{aligned} \tag{2.99}$$

for all $n \in \mathbb{Z}$. Equivalently, the spatially-invariant systems Φ^x, Φ^u are of the form (2.57), (2.58). Conversely if Φ^x, Φ^u are of the form (2.57), (2.58), then $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ so that K is stabilizing.

2.7.2 Proof of Theorem 2.5.4:

A direct application of the results of [20] show that the spatially-invariant system K is a stabilizing controller for (2.69) if and only if the resulting closed-loop mappings $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ and satisfy the affine constraint:

$$\left((sI + I) - (A+I)\right)\Phi^x - B_2\Phi^u = I, \tag{2.100}$$

where A and B_2 denote pointwise the multiplication operators defined by the finite-dimensional matrices A and B_2. The affine constraint (2.100) can be written equivalently in terms of the convolution kernels of $\Phi^x = \{\phi_m^x\}_{m\in\mathbb{G}}$ and $\Phi^u = \{\phi_m^u\}_{m\in\mathbb{G}}$ as:

$$\begin{bmatrix} \left((s+1)I - (A+I)\right) & -B_2 \end{bmatrix} \begin{bmatrix} \phi_m^x(s) \\ \phi_m^u(s) \end{bmatrix} = I, \ \forall m \in \mathbb{G} \tag{2.101}$$

and $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ if and only if $\Phi_m^x, \Phi_m^u \in \overline{\mathcal{R}}_s$ for all $m \in \mathbb{G}$. By Theorem 2.5.3, each ϕ^x and ϕ^u satisfy (2.101) if and only if each ϕ_m^x and ϕ_m^u is of the form (2.70). Equivalently, Φ^x, Φ^u are of the form (2.71).

2.7.3 Proof of Lemma 2.5.5:

We write the problem (2.76) in model-matching form (2.78) as

$$J = \inf_{Q \in \overline{\mathcal{R}}_s} \; \|H + UQ\|^2_{\mathcal{H}_2}$$

$$:= \inf_{\theta \in \overline{\mathcal{R}}_s} \left\| \begin{bmatrix} \frac{1}{s+1} \\ \frac{-1}{s+1} \\ 0 \\ 0 \\ \hline \frac{-\gamma}{s+1} \\ 0 \\ 0 \end{bmatrix} + \frac{1}{s+1} \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \\ 1 & 0 & 0 \\ \hline 0 & \gamma s & 0 \\ 0 & 0 & \gamma s \\ \gamma s & 0 & 0 \end{bmatrix} \begin{bmatrix} \theta_{M\text{-}1} \\ \theta_0 \\ \theta_1 \end{bmatrix} \right\|^2_{\mathcal{H}_2} \tag{2.102}$$

We compute an inner-outer factorization $U = U_i U_o$ [31] as follows. U_o is given as a spectral factor of $U^\sim U$:

$$\begin{aligned} U^\sim U &= \frac{1}{(s+1)(-s+1)} \left(T + -\gamma^2 s^2 I\right) \\ &= \frac{1}{(s+1)(-s+1)} \left(V\Lambda V^T + -\gamma^2 s^2 I\right) \\ &= \frac{1}{(-s+1)} V(\Lambda^{1/2} - \gamma s I) \cdot \frac{1}{s+1}(\Lambda^{1/2} + \gamma s I)V^* \quad =: U_o^\sim U_o, \end{aligned}$$

where $V\Lambda V^T$ is an eigenvector decomposition of $T := \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$. Then U_i is given by $U_i := UU_o^{-1}$, and optimal solution of (2.102) is given by

$$Q = \begin{bmatrix} \theta_{-1} & \theta_0 & \theta_1 \end{bmatrix}^T = U_o^{-1} \left(U_i^\sim H\right) \Big|_{\overline{\mathcal{R}}_s}. \tag{2.103}$$

The corresponding optimal closed-loop norm is:

$$J = \left\| \left(U_i^\sim H\right) \Big|_{\mathcal{RH}_2^\perp} \right\|^2_{L_2} \tag{2.104}$$

where $U_o^{-1} \left(U_i^\sim H\right) \Big|_{\overline{\mathcal{R}}_s}$ is the projection of $U_o^{-1} \left(U_i^\sim H\right)$ onto $\overline{\mathcal{R}}_s$ and $\left(U_i^\sim H\right) \Big|_{\mathcal{RH}_2^\perp}$ is the projection of $(U_i^\sim H)$ onto $\mathcal{RH}_2^\perp$, the orthogonal complement of $\overline{\mathcal{R}}_s$ in the space of rational functions in $L^2(j\mathbb{R})$. We compute $U_i^\sim H$ as

$$U_i^\sim H = \frac{1}{\sqrt{2}(s+1)} \cdot \begin{bmatrix} \frac{2-\sqrt{2}+\gamma^2 s}{(2-\sqrt{2})^{1/2} - \gamma s} \\ 0 \\ \frac{2+\sqrt{2}+\gamma^2 s}{(2+\sqrt{2})^{1/2} - \gamma s} \end{bmatrix},$$

and the projections of $U_i^\sim H$ onto $\overline{\mathcal{R}}_s$ and $\mathcal{RH}_2^\perp$ are computed using partial fraction expansions. Expressions for the optimal Θ and closed-loop norm then follow from equations (2.103) and (2.104), respectively.

2.7.4 Computations for First Order Consensus Problem

We analytically compute the optimal Θ which solves (2.76) for each of the following measures and band size constraints:

- **Local Error, Band Size 2**

$$\theta_0 = \frac{-1}{\sqrt{3}}\left(\frac{\alpha_1}{(2-\sqrt{3})^{1/2}+\gamma s}+\frac{\alpha_2}{\sqrt{2}+\gamma s}+\frac{-\alpha_3}{(2+\sqrt{3})^{1/2}+\gamma s}\right)$$

$$\theta_1 = \theta_{-1} = \frac{-1}{2}\left(\frac{\alpha_1}{(2-\sqrt{3})^{1/2}+\gamma s}+\frac{\alpha_3}{(2+\sqrt{3})^{1/2}+\gamma s}\right)$$

$$\theta_2 = \theta_{-2} = \frac{1}{2\sqrt{3}}\left(\frac{-\alpha_1}{(2-\sqrt{3})^{1/2}+\gamma s}+\frac{2\alpha_2}{\sqrt{2}+\gamma s}+\frac{\alpha_3}{(2+\sqrt{3})^{1/2}+\gamma s}\right)$$

where $\alpha_1 := \frac{\gamma-(2-\sqrt{3})^{1/2}}{\sqrt{3}}$, $\alpha_2 := \frac{\gamma-\sqrt{2}}{\sqrt{3}}$, and $\alpha_3 := \frac{-\gamma+(2+\sqrt{3})^{1/2}}{\sqrt{3}}$.

- **Deviation from Average, Band Size 1**

$$\theta_0 = \frac{2(\gamma-1)}{3(1+\gamma s)} - \frac{\sqrt{1-\frac{3}{N}}-\gamma}{3\left(\sqrt{1-\frac{3}{N}}+\gamma s\right)}$$

$$\theta_1 = \theta_{-1} = \frac{\gamma-1}{3(1+\gamma s)} - \frac{\sqrt{1-\frac{3}{M}}-\gamma}{3\left(\sqrt{1-\frac{3}{M}}+\gamma s\right)}$$

- **Deviation from Average, Band Size 2**

$$\theta_0 = \frac{-1}{4}\cdot\frac{1-\gamma}{1+\gamma s} + \frac{\gamma-\sqrt{1-\frac{5}{M}}}{5\left(\sqrt{1-\frac{5}{M}}+\gamma s\right)}$$

$$\theta_1 = \theta_{-1} = \frac{1}{5}\cdot\frac{1-\gamma}{1+\gamma s} + \frac{\gamma-\sqrt{1-\frac{5}{M}}}{5\left(\sqrt{1-\frac{5}{M}}+\gamma s\right)}$$

$$\theta_2 = \theta_{-2} = \theta_1 = \theta_{-1}$$

We note that the optimal closed-loop mappings can be recovered from Θ through the formula (2.81).

2.7.5 Completion of Proof of Theorem 2.5.8:

Let $u = Kx$ be a (dynamic or static) controller for (2.91) and let Φ^x, Φ^u denote the corresponding closed-loop mappings. If Φ^x, Φ^u are of the forms (2.94) and (2.95) for some $\theta \in \overline{\mathcal{R}}_s$, then they are both elements of $\overline{\mathcal{R}}_s$ and direct computations show that (2.105) holds. Conversely, assume that K is stabilizing. Then $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ and following [11], direct computations show that Φ^x and Φ^u satisfy

$$(sI - A)\Phi^x(s) = I + B_2\Phi^u(s), \tag{2.105}$$

where $A = \text{diag}\{a_m\}$, $B_2 = \text{diag}\{b_m\}$. Equivalently,

$$\phi^x_{ij}(s) = \begin{cases} \frac{b_i}{s-a_i}\phi^u_{ij}(s), & i \neq j \\ \frac{1}{s-a_i} + \frac{b_i}{s-a_i}\phi^u_{ij}(s), & i = j \end{cases} \tag{2.106}$$

where ϕ^x_{ij} and ϕ^u_{ij} are the (i,j) components of Φ^x and Φ^u. If $\Re(a_i) \geq 0$, then a necessary condition for Φ^u to be stable is that ϕ^u_{ij} does not have a pole at a_i for any j. Then, since ϕ^u_{ij} is strictly proper, it must be of the form:

$$\phi^u_{ij}(s) = \begin{cases} \frac{s-a_i}{s+1}\theta_{ij}(s) & i \neq j \\ \frac{1}{b_i}\left(\frac{s-a_i}{s+1}\theta_{ii}(s) - \frac{a_i+1}{s+1}\right), & i = j \end{cases} \tag{2.107}$$

which is equivalent to the parameterization provided in (2.95). Substituting (2.107) into (2.106) shows that Φ^x is of the form (2.94).

2.7.6 Proof of Theorem 2.5.9:

By Lemma 2.4.3, it is sufficient to show that $\Phi^x, \Phi^u \in \overline{\mathcal{R}}_s$ satisfy the affine constraint (2.30) if and only if they are of the form (2.63). The structure of A and B_2 allow the constraint (2.30) to be written componentwise as:

$$(A^{(i)} + I)\phi^x_{ij} - B_2^{(i)}\phi^u_{ij} = \begin{cases} I, & i = j \\ 0, & i \neq j \end{cases} \tag{2.108}$$

For the case $i = j$, parameterizations of ϕ^x_{ii} and ϕ^u_{ii} are then given by Theorem 2.5.3. A back-stepping approach similar to that used in the proof of Theorem 2.5.3 can be used to derive a parameterization of ϕ^x_{ij} and ϕ^u_{ij} for the case $i \neq j$. The details of this procedure are omitted. ■

Chapter 3

Controller Structure vs. Closed-Loop Structure of Spatially-Distributed Systems

Abstract* - We address the optimal distributed controller design problem, imposing locality constraints to account for subcontroller interaction restricted to local neighborhoods as specified by an underlying graph structure. We provide a detailed characterization of such locality constraints imposed on an implementation of a controller (sparsity of state space matrices) in contrast to locality constraints on the sparsity of the controller input-output mapping (transfer function). The set of controllers with local implementations is in general non-convex and complex to characterize. Constraints on the sparsity of the closed-loop transfer function however are convex, and an implementation of the corresponding controller inherits this structure. Thus, the closed-loop structured transfer function design problem provides a convex relaxation of the structured implementable controller design problem. We take a first step toward quantifying the performance gap between these problems by focusing in on the *relative feedback* setting. Our main result demonstrates that when such additional relative feedback constraints are imposed, this convex relaxation may be *infeasible.

This Chapter is based on the following Publications:

[32] -E. Jensen and B. Bamieh, On the Gap Between System Level Synthesis and Structured Controller Design: The Case of Relative Feedback, in 2020 Annual American Control Conference (ACC), pp. 4594–4599, IEEE, 2020.

[33] - E. Jensen and B. Bamieh, On Structured-Closed-Loop versus Structured-Controller Design: the Case of Relative Measurement Feedback, IEEE Transactions on Automatic Control (Submitted).

3.1 Introduction

We consider the design of controllers for spatially distributed systems. The controller to be designed is composed of many spatially distributed subcontrollers, each with access to a limited subset of *local* information. Subcontrollers are restricted to communicate this local information to only a limited *local* subset of neighboring subsystems, leading to constrained controller design problems of the form:

$$\begin{aligned} \inf_{K} \ & \|\mathcal{F}(P;K)\| \\ \text{s.t.} \ & \text{Subcontroller communication constraints,} \end{aligned}$$

with $\mathcal{F}(P;K)$ the closed-loop feedback interconnection between a generalized plant P and controller K [26]. Throughout this paper, we will formally define various precise notions of such subcontroller communication constraints.

Previous work has largely focused on enforcing information sharing constraints via a structural sparsity constraint on the controller transfer matrix as

$$\begin{aligned} \inf_{K} \ & \|\mathcal{F}(P;K)\| \\ \text{s.t.} \ & K(s) \text{ structured,} \end{aligned}$$

however, in general this problem is non-convex. Recent works suggest the importance of looking beyond structured transfer matrices, and consider the structure of a state-space realization used to implement a controller instead [12], [14], [13]. Indeed, a transfer matrix K may have a realization composed of sparse matrices, even if K itself is not sparse.

However, characterizing the set of controllers which have structured implementations remains an open problem. A primary reason for this is that the state space realizations of a given controller vary in structure (sparsity pattern of state matrices) and in number of states (dimensions of state matrices). Classifying the existence of one realization with a prescribed structure is unwieldy, although recent works have provided preliminary results [12–14]. In addition, the System Level Synthesis (SLS) framework [11] has suggested an alternate approach by directly designing the closed-loop mappings, as opposed to the controller transfer matrix or corresponding Youla parameter. It is known that the set of closed-loop mappings is convex [15], and adding additional convex structural constraints will preserve convexity. Such constraints do not imply that the controller transfer function itself will have this same sparsity pattern but ensure that the corresponding controller will have an *implementation* that inherits this prescribed structure.

To understand the usefulness of the SLS method, we compare the optimal performance of a structured closed-loop transfer function to that of a structured implementable controller. We formally demonstrate that the optimal controller design problem subject to sparsity constraints on the closed-loop transfer functions (solved for via SLS) provides a convex relaxation of the optimal controller design problem subject to locality constraints on controller implementations.

$$\begin{array}{llcll} \inf\limits_{K} & \|\mathcal{F}(P;K)\| & \leq & \inf\limits_{K} & \|\mathcal{F}(P;K)\| \\ \text{s.t.} & K(s) \text{ structured} & & \text{s.t.} & \text{closed-loop structured,} \end{array}$$

We begin to quantify this performance gap by analyzing the *relative feedback* control design problem from the SLS perspective. Such relative feedback constraints are natural in e.g. consensus and formation control problems, and are reasonable when the underlying plant itself has relative dynamics. We remark that although relative design requirements are implicit in many consensus like algorithms, few works have included this as an explicit design constraint. The unconstrained LQR design problem for a system with relative dynamics (and with relative LQR weighting matrices) is *static* and given by a *relative* matrix obtained using the algebraic Riccati equation. With additional locality constraints however, this remains an open problem in general. [23] provided a solution to this optimal relative and structured controller design problem in the case that the controller is restricted to be static. We note that the techniques employed in [23] cannot be utilized to analyze controllers with arbitrary local degree, and an alternate tractable method for this more general setting has yet to be developed.

We employ an example to demonstrate our main results on the usefulness of SLS in this relative feedback setting, We consider the optimal control design problem for a distributed plant composed of N 1^{st}-order subsystems on the undirected torus $\mathbb{Z}_N$ with performance output capturing a measure of consensus, subject to relative feedback and closed-loop spatial spread constraints. We demonstrate that when this consensus measure is defined in terms of a matrix of rank greater than the prescribed closed-loop spatial spread, there does not exist a feasible solution to the constrained $\mathcal{H}_2$ controller design problem. On the other hand, stabilizing, relative controllers with structured implementations for this problem are easily constructed. This highlights that although closed-loop transfer function sparsity is sufficient to ensure a structured implementation, this requirement may be far from necessary in the relative feedback setting. We remark that this result does not contradict any of the results presented in Chapter 2, as Chapter 2 did not incorporate any relative feedback requirements.

The remainder of this chapter is structured as follows. In Section 3.2, we formally present three notions of locality constraints on distributed controller design: structured-realizable, network-realizable, and closed-loop transfer-function-structured. We introduce the structured-realizable optimal controller design problem as well as the convex relaxation of this problem provided by SLS. In Section 3.3, we demonstrate that this convex relaxation may not be feasible when relative feedback constraints are imposed. We analyze the performance gap between the optimal structured-realizable and optimal closed-loop structured controllers in the relative feedback setting in Section 3.4. Section 3.5 provides generalizations to higher spatial dimensions.

3.2 Problem Formulation

We set up the control problem in the framework of the "standard problem of robust control" with

$$\begin{bmatrix} z \\ y \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \begin{bmatrix} w \\ u \end{bmatrix} \tag{3.1}$$

where the vectors w, u, y and z are the exogenous disturbances, control signals, controller measurements and performance output respectively. The system P is referred to as the generalized plant. In a spatially distributed system, the signal vectors w, u and y are partitioned into local sub-signals as

$$u = \begin{bmatrix} u_1^T & \cdots & u_N^T \end{bmatrix}^T, \tag{3.2}$$

where u_i is the control signal *at the i'th site*, and similarly for w and y. The performance output z however may contain global objectives, and therefore may not be similarly partitioned. We will often work with state space realizations of the generalized plant P which we assume to be of the form

$$P = \left[\begin{array}{c|cc} A & B_1 & B_2 \\ \hline C_1 & 0 & D_{12} \\ C_2 & D_{21} & 0 \end{array}\right] = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}. \tag{3.3}$$

The state signal x is partitioned similarly to (3.2). Note that the partitioning of signals induces partitionings of the realization matrices as well.

When the feedback loop is closed with controller $u = Ky$, the closed loop system is denoted by

$$\mathcal{F}(P;K) := P_{11} + P_{12}K\,(I - P_{22}K)^{-1}\,P_{21}.$$

The usual robust controller design problem is to find controllers that minimize some closed-loop norm $\|\mathcal{F}(P;K)\|$. In distributed control problems however, there are usually additional requirements of *controller locality* which encode constraints about which site measurements the control signal for each site can depend on. In this chapter, we also consider an additional requirement of *relative measurements* which is typically not explicitly stated in distributed control design problems, but is implicit in many consensus-like designs. The two requirements of locality and relative measurements are unrelated to each other in that one can demand either one, or both. In this chapter we consider requiring both, and we formalize these two notions in the next two subsections.

3.2.1 Locality Constraints

A common requirement in structured controller design is to restrict sub-controllers to have access to only a *local* subset of measurements. We specify sub-controller communication requirements in terms of a graph with adjacency matrix $\mathcal{A}$ defined by $\mathcal{A}_{ij} = 1$ if there exists an edge between nodes i and j, and $\mathcal{A}_{ij} = 0$ otherwise. We refer to a graph and its adjacency matrix synonymously. In addition, we will need the "b-hops" graph, which is defined by

$$\mathcal{A}_{ij}^{(b)} = \begin{cases} 1, & \text{if } \left(\mathcal{A}^b\right)_{ij} \neq 0 \\ 0, & \text{else,} \end{cases} \tag{3.4}$$

where $\mathcal{A}^b$ is the b'th power of $\mathcal{A}$. Thus $\mathcal{A}^{(b)}$ is the adjacency matrix of a graph where an edge between nodes i and j is present if there is a path of length $\leq b$ between those nodes.

Locality of Realizations

In this paper, we consider controllers that can be *realized* with structured matrices. We state this formally.

Definition 3.2.1 *Consider a graph $\mathcal{A}$ with N nodes and a matrix M partitioned into $N \times N$ blocks. We say that M is* graph-structured *(or $\mathcal{A}$-*structured*) if the ij'th block of M is zero whenever $\mathcal{A}_{ij} = 0$. Thus the only non-zero blocks in M correspond to edges in the graph. In this case we use the notation $M \in \mathcal{S}(\mathcal{A})$.*

Definition 3.2.2 *An LTI system H is* structured-realizable *with respect to a graph $\mathcal{A}$ ($\mathcal{A}$-structured-realizable) if there exists a realization with graph-structured matrices, i.e.*

$$H = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right], \qquad A, B, C, D \in \mathcal{S}(\mathcal{A}).$$

If in addition either B, D or C, D are block diagonal, then we call H network-realizable.

These notions are easily extended to LTV systems.

Definition 3.2.3 *An LTV system H is* structured-realizable *with respect to a graph $\mathcal{A}$ if there exists a (time-varying) realization $H = \left[\begin{array}{c|c} A(t) & B(t) \\ \hline C(t) & D(t) \end{array}\right]$ for which $A(t), B(t), C(t), D(t) \in \mathcal{S}(\mathcal{A})$ for all t. If in addition, $B(t), D(t)$ or $C(t), D(t)$ are block diagonal for all time, then H is* network-realizable.

The motivation for this definition is that a structured-realizable system (e.g. a controller) can be realized with differential equations that require information from sites within a local neighborhood to compute a given local output. Note that the statement $A, B, C, D \in \mathcal{S}(\mathcal{A})$ automatically implies that inputs, outputs and states are partitioned as in (3.2).

Network realizability is a stronger requirement, originally introduced in [14]. It characterizes systems in which the output of any node depends on the states of neighboring nodes through a transfer function of relative degree of at least one (in the case where one requires C, D to be block diagonal). In discrete time, this means that there is at least a one-step delay in transmitting a node's state to its neighbors. In continuous time, it implies that such transmission does not happen instantly, but rather through a strictly proper transfer function. The class of network-realizable systems is closed under additions, cascades and feedbacks, while the class of structured-realizable systems is not. Note that a network-realizable system is structured-realizable, but the converse is not necessarily true.

Remark 3 *A structured realization (or a network realization) is likely non-minimal, with the size of each local state determined by the dimension of the blocks composing the "A" matrix. Classifying the existence of a structured state-space realization that is also stabilizable and detectable remains an open problem in general, although recent work [12, 13] has provided solutions for certain subclasses of systems.*

Locality of Transfer Functions

Unlike the concept of locality of realizations, which has only very recently been studied [12,14], the concept of local transfer functions (or input-output relations) has a longer history. We define it formally next.

Definition 3.2.4 *An $N \times N$ block-partitioned LTI system H is called* Transfer-Function Structured *(*TF-structured*) with respect to a graph $\mathcal{A}$ if*

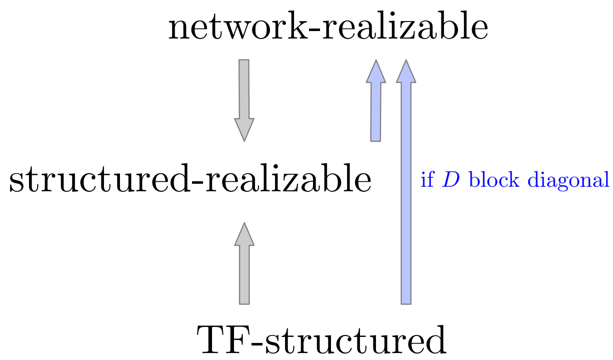
$$H_{ij}(s) = 0, \quad \text{when } \mathcal{A}_{ij} = 0,$$

where H_{ij} is the ij'th block of the transfer function matrix. In other words, the block sparsity structure of H is the same as the adjacency matrix $\mathcal{A}$.

A structured-realizable system may not be TF-structured, but any TF-structured system is structured-realizable (with the same structure), as stated in the following lemma, whose proof is in Appendix 3.7.1. We note that similar results relating the structure of a transfer function to the structure of a corresponding state-space realization have appeared in e.g. [12]. In the sequel, whenever the phrases *structured-realizable*, *network-realizable*, or *TF-structured* are used, it is assumed that there is an underlying graph $\mathcal{A}$ that refers to those structures, and $\mathcal{A}$ will not be explicitly mentioned when no confusion can occur.

Lemma 3.2.1 *If H is TF-structured, then H is structured-realizable. If in addition H has a block-diagonal feedthrough term, then it is also network-realizable. The converse does not hold: there exist structured-realizable (and network-realizable) systems H which are not TF-structured.*

A canonical example that helps to appreciate the above statement is a line-graph network with nearest neighbor interactions. The structured matrices in this case are the tri-diagonal matrices. The transfer function $(sI - A)^{-1}$ with A tri-diagonal will in general not be tri-diagonal, and in fact will be a dense matrix. The implications in the above lemma are summarized in the following diagram.



It follows directly from Lemma 3.2.1, that the TF-structured optimal controller design problem provides an upper bound on the structured-realizable controller design problem. In particular,

$$\begin{array}{ll} \inf\limits_{K} & \|\mathcal{F}(P;K)\| \\ \text{s.t.} & K \text{ is structured-realizable} \end{array} \quad \leq \quad \begin{array}{ll} \inf\limits_{K} & \|\mathcal{F}(P;K)\| \\ \text{s.t.} & K \text{ is TF-structured,} \end{array}$$

and

$$\begin{array}{ll} \inf\limits_{K} & \|\mathcal{F}(P;K)\| \\ \text{s.t.} & K \text{ is network-realizable} \end{array} \quad \leq \quad \begin{array}{ll} \inf\limits_{K} & \|\mathcal{F}(P;K)\| \\ \text{s.t.} & K \text{ is TF-structured} \\ & \lim\limits_{s\to\infty} K(s) \text{ block diagonal,} \end{array}$$

where $\|\mathcal{F}(P;K)\|$ denotes some closed-loop norm. It is clear that for system implementations with only local interactions, the concepts of network or structured realizability are the important ones. However, to impose that directly as a design constraint appears to be unwieldy. Imposing TF-structured constraints is more tractable (in certain problem settings), and by the above lemma, it is one way (though a conservative one) to arrive at the ultimate goal, which is an implementable realization in a networked setting.

3.2.2 Relative Measurements

A separate (from locality) requirement on the controller structure is for it to be constrained to operate on *relative measurements*. Such constraints arise naturally in control problems where only differential sensors are available, such as in vehicular formations with ranging measurement devices, or mechanical control problems where only relative strain measurements are possible. We define this notion formally as follows.

Definition 3.2.5 *Consider signals u and y partitioned into N sub-signals as in (3.2), and a transfer function (or real-valued) matrix K partitioned conformably with $u = Ky$. The matrix K is called* relative *if each component u_n of the output can be written as a function of only differences of inputs, i.e.*

$$u_n = \sum_{1\leq i<j\leq N} \mathcal{K}_{ij}^n \, (y_i - y_j), \tag{3.5}$$

where each $\mathcal{K}_{ij}^n$ is some transfer function matrix.

Note that $\mathcal{K}_{ij}^n$ is not the ij'th entry of the transfer function matrix K, but rather the transfer function that acts on the relative measurement $y_i - y_j$ to produce the output u_n. As shown in Appendix 3.7.2, the representation (3.5) is non-unique, but there is a compact characterization of when a transfer function matrix is relative.

Lemma 3.2.2 *A transfer function matrix K is relative if and only if*

$$K(s)\mathbb{1} = 0,$$

where $\mathbb{1}$ is the vector with all entries of 1.

Proof: There are many ways to take a matrix that satisfies $K\mathbb{1} = 0$ and rewrite the relation $u = Ky$ in a form like (3.5). Thus, the form (3.5) is non-unique. These constructions and the proof of this Lemma are detailed in Appendix 3.7.2. ■

Notice the similarity with the condition for a right stochastic matrix, except that here K can be a transfer function matrix, and has no positivity constraints.

3.2.3 Structured-Realizable and Relative Controllers

We now formally state the structured-realizable controller design problem we are concerned with. In this paper we use the $\mathcal{H}_2$ norm as the performance measure, although this problem can be stated in the same manner using any other system norm. The objective is to design a (static or dynamic) feedback controller $u = Ky$ that uses only relative measurements, and also satisfies locality constraints according to the structure of a graph $\mathcal{A}$. We reiterate that these are two additional requirements on traditional controller design that can be imposed independently of each other.

Structured-Realizable, Relative Controller Design:
Given a graph $\mathcal{A}$, and a plant P (3.1) with signals partitioned according to $\mathcal{A}$, find

$$\left.\begin{array}{lll} \inf\limits_{K} & \|\mathcal{F}(P;K)\|_{\mathcal{H}_2} & \\ \text{s.t.} & K \text{ is structured-realizable} & \text{(locality)} \\ & K\mathbb{1} = 0 & \text{(relative feedback)} \end{array}\right\} \qquad (3.6)$$

Note that Lemma 3.2.2 was used to state the relative measurements constraint compactly. The problem (3.6) has a finite optimum only if there exists an $\mathcal{F}(P;K)$ satisfying the constraints with $\|\mathcal{F}(P;K)\|$ finite. Otherwise the optimal value of (3.6) is infinite and the problem is said to be *infeasible*.

This problem as stated even without the relative measurements assumption is typically non-convex, and a general solution remains elusive. However, many consensus-type algorithms in the literature can be viewed as upper bounds on this problem. The algorithm we present in Section 3.3 is such a bound. In addition, the relative measurements requirement amounts to a type of "conservation law" if the plant P itself satisfies a similar requirement.

Note that by Lemma 3.2.1, the optimal TF-structured controller design problem (obtained by replacing the constraint that K be $\mathcal{A}$-structured-realizable in (3.6) with the constraint that K be TF-structured) provides an upper bound on the structured-realizable controller design problem (3.6). Even so, the TF-structured design problem is typically non-convex except when the graph structure satisfies properties such as "funnel causality" [8] or "quadratic invariance" [7]. We are interested in a wider class of problems in this chapter for which there is no currently obvious convex problem re-formulation.

3.2.4 Closed-loop Design

Instead of directly constraining the structure of the controller (either of its realization or its transfer function), one can indirectly obtain structured controllers (though not all of them) by constraining the *closed loop* instead. It is known that the set of closed-loops corresponding to all possible stabilizing controllers is an affine linear set [15]. Obviously, the closed-loop design problem remains convex if further convex constraints are imposed on the closed loop, e.g. convex locality constraints. If one can recover the controller from the closed-loop design in a way that inherits the closed-loop structural constraints, then this is at least one method to relax the above problem to a convex one. This is the theme followed by the System Level Synthesis (SLS) framework [11], which we refer to in this chapter as *structured-closed-loop* design. We summarize briefly the two main ideas behind this procedure.

Consider the plant (3.3) in feedback with a (dynamic or static) controller $u = Kx$. The following is a summary of the SLS framework. The reader should examine Figure 3.1 for the descriptions of the various systems.

- The original control design problem for P is reformulated as one for $\tilde{P}$ with an equivalent objective

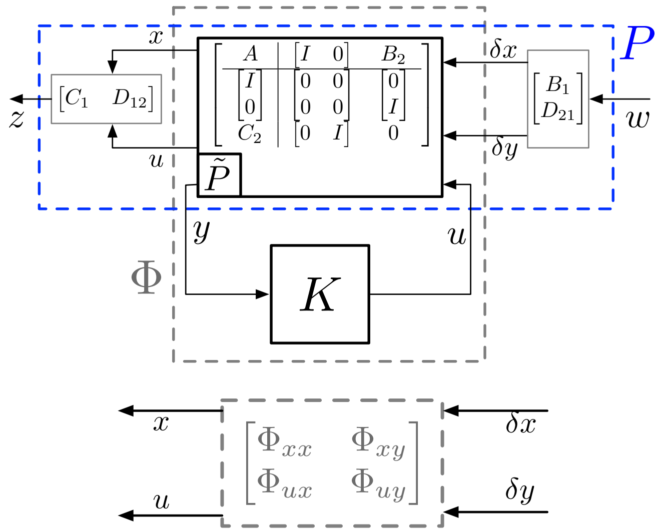
$$\inf_K \|\mathcal{F}(P;K)\| \;=\; \inf_K \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \mathcal{F}\left(\tilde{P};K\right) \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \right\| \tag{3.7}$$

- Characterizing feasible closed loops $\mathcal{F}\left(\tilde{P};K\right)$ is much simpler than those of $\mathcal{F}(P;K)$

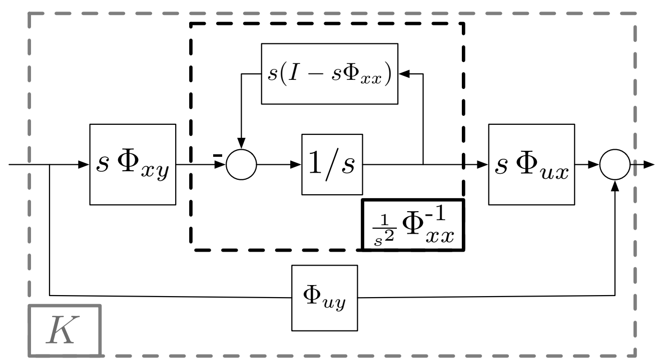
$$\Phi = \begin{bmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} = \mathcal{F}\left(\tilde{P};K\right) \text{ for some } K$$

$$\Leftrightarrow \left\{ \begin{array}{l} \begin{bmatrix} sI\text{-}A & \text{-}B_2 \end{bmatrix} \begin{bmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} = \begin{bmatrix} I & 0 \end{bmatrix} \\ \begin{bmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{ux} & \Phi_{uy} \end{bmatrix} \begin{bmatrix} sI\text{-}A \\ \text{-}C_2 \end{bmatrix} = \begin{bmatrix} I \\ 0 \end{bmatrix} \end{array} \right. \tag{3.8}$$

- Given a closed loop Φ, the corresponding controller $K = \Phi_{uy} - \Phi_{ux}\Phi_{xx}^{\text{-}1}\Phi_{xy}$ can be implemented with the diagram in Figure 3.1b. This implementation mirrors any structural constraints imposed on Φ.



(a) Instead of characterizing the affine linear set of all closed loops $\mathcal{F}(P;K)$ with complicated interpolation constraints, one can characterize the closed loops $\Phi = \mathcal{F}(\tilde{P};K)$, for which the affine linear constraints (3.8) are much simpler. The design problem is then in terms of Φ, but the objective function remains the original one.



(b) The controller $K = \Phi_{uy} - \Phi_{ux}\Phi_{xx}^{-1}\Phi_{xy}$ can be recovered from the closed loop Φ. It can be implemented with the diagram above so that the spatial sparsity structure of Φ is preserved in each of the blocks above.

Figure 3.1: The two key ideas in the SLS framework is to (*top*) characterize a different closed loop mapping than $\mathcal{F}(P;K)$, and then (*bottom*) implement the controller K from the closed loop maps Φ in a manner that mirrors any structural constraints imposed on the closed loop design.

There are two key ideas in this framework. The first is that the constraints (3.8) on $\Phi = \mathcal{F}(\tilde{P};K)$ are much simpler than the interpolation constraints (involving MIMO zeros and their directions) that would be needed on $\mathcal{F}(P;K)$ in general. The second idea is that if any type of convex structural constraints, such as locality, are imposed on the closed loop Φ, then an implementation like Figure 3.1b preserves that locality structure in the implementation of K. We call such controllers *closed-loop structured*.

Definition 3.2.6 *Given a plant $\tilde{P}$ (Figure 3.1a), a controller K is called* closed-loop-transfer-function structured *(closed-loop TF-structured) with respect to graph $\mathcal{A}$ if it results in a closed loop $\Phi = \mathcal{F}(\tilde{P}; K)$ that is TF-structured.*

Remark 4 *Closed-loop TF-structured constraints (Definition 3.2.6) extend the closed-loop band-size constraints introduced in Chapter 2 (Definition 2.2.2) to apply to more general underlying graph structures.*

This notion of closed-loop TF-structure is implicitly defined in terms of the plant to be controlled since the closed-loop transfer functions are defined in terms of the plant parameters A, B_2, C_2. Thus, a controller may be closed-loop TF-structured for one plant but not for another. In contrast, the notions of structured-realizability and network-realizability are explicit properties of the controller that are independent of choice of plant. The following theorem provides a relation between the set of structured-realizable controllers and the set of closed-loop TF-structured controllers for a given plant.

Theorem 3.2.3 *If a proper transfer function K is a closed-loop TF-structured controller for any plant P with respect to a graph $\mathcal{A}$, then K is an $\mathcal{A}$-structured-realizable controller.*

Proof: To prove this result, we leverage the fact that a controller can be implemented directly using the resulting closed-loop transfer functions, preserving the closed-loop structure (see Figure 3.1b). We employ Lemma 3.2.1 to demonstrate that this implementation leads to an $\mathcal{A}$-structured-realizable controller K. The details of the proof of Theorem 3.2.3 are provided in Appendix 3.7.3. ■

As shown by Theorem 3.2.3, the SLS design procedure yields a structured-realizable controller; this demonstrates that optimization over the set of closed-loop TF-structured controllers for plant P provides an upper bound on optimization over the set of structured-realizable controllers (3.6). This TF-closed-loop structured design problem is formally stated as follows.

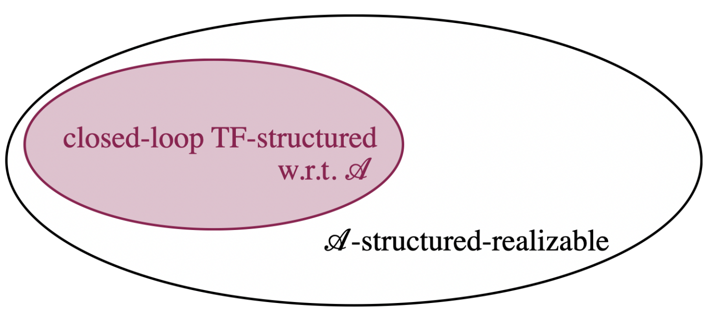


Figure 3.2: The set of controllers which are closed-loop TF-structured for any plant P is a subset of the set of structured-realizable controllers

TF-Structured-Closed-Loop, Relative Controller Design:

Given a graph $\mathcal{A}$, and a plant P (3.1) with signals partitioned according to $\mathcal{A}$, find

$$\left.\begin{array}{ll} \inf\limits_{K} & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \mathcal{F}(\tilde{P};K) \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \right\|_{\mathcal{H}_2} \\ \text{s.t.} & \Phi = \mathcal{F}(\tilde{P};K) \text{ is TF-structured} \qquad \text{(locality)} \\ & K\mathbb{1} = 0 \qquad \text{(relative feedback)} \end{array}\right\} \tag{3.9}$$

In the next section we will see that the relative feedback constraint is actually convex on the closed loop $\Phi = \mathcal{F}(\tilde{P};K)$ in certain problem settings as well. Thus, optimization over the set of TF-closed-loop structured controllers is convex [11], and we therefore say that the problem (3.9) is a convex relaxation of (3.6).

In this chapter we are primarily concerned with contrasting the structured-realizable controller design problem (3.6) with the TF-structured-closed-loop problem (3.9). These could also be compared to the widely studied problem of TF-structured controller design, and we just briefly mention this third problem here. The optimal controller design subject to sparsity constraints imposed directly on the controller transfer function directly is generally intractable, except for classes of structures of the so-called funnel causality [8] or quadratic invariance [7] types. For such classes of structures, there is a one-to-one correspondence between TF-structured and closed-loop TF-structured controllers, and therefore the two problems are equivalent. For general structured classes however, We emphasize that there is no equivalence (either as lower or upper bounds) between the TF-structured and closed-loop TF-structured controller design. An example in Appendix 3.7.4 illustrates this point.

3.3 Main Result

Our main result identifies potential limitations of the SLS design procedure with TF-structured constraints in the relative feedback setting. In particular, we demonstrate through an example that there may not exist a feasible solution to the relative feedback, TF-structured closed-loop $\mathcal{H}_2$ design problem (3.9). We will later comment on alternative possibilities for closed-loop structural constraints, as well as the special issues that arise when imposing relative feedback as a design constraint.

The example we consider is a state-feedback design problem. Thus, we first provide a presentation of the closed-loop structured design problem in the state-feedback setting. For a more complete review of state-feedback SLS, we refer the reader to [11].

3.3.1 State Feedback Closed-loop Design

Consider a (dynamic or static) state feedback controller $u = Kx$ in feedback with a plant of the form

$$\begin{aligned} \dot{x} &= Ax + B_1 w + B_2 u \\ z &= C_1 x + D_{12} u. \end{aligned} \tag{3.10}$$

The closed-loop mapping from disturbance w to performance output z can be written as

$$\begin{aligned}\mathcal{F}(P;K) &= \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} I \\ K(s) \end{bmatrix} (sI - A - B_2 K(s))^{-1} B_1 \\ &= \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi_x(s) \\ \Phi_u(s) \end{bmatrix} B_1. \end{aligned} \tag{3.11}$$

$\Phi_x(s)$ and $\Phi_u(s)$ are the transfer functions from $B_1 w$ to state x and control action u of plant P in feedback with controller K, and are referred to as the closed-loops. The following lemma provides a complete characterization of all feasible closed-loops for system (3.3).

Lemma 3.3.1 *$u = Kx$ is a state-feedback controller for* (3.3) *if and only if the resulting closed-loops are strictly proper transfer functions, and the affine relation*

$$f(s) := \begin{bmatrix} sI - A & -B_2 \end{bmatrix} \begin{bmatrix} \Phi_x(s) \\ \Phi_u(s) \end{bmatrix} = I \tag{3.12}$$

holds for all s except possibly isolated points (removable singularities of f). The controller $u = Kx$ can be recovered from the closed-loops as

$$u = Kx = \Phi_u (\Phi_x)^{-1} x.$$

Moreover, K can be implemented as

$$\begin{aligned} v &= x + (I - s\Phi_x(s))v \\ u &= s\Phi_u(s)v, \end{aligned} \tag{3.13}$$

preserving the closed-loop structure (see Figure 3.3).

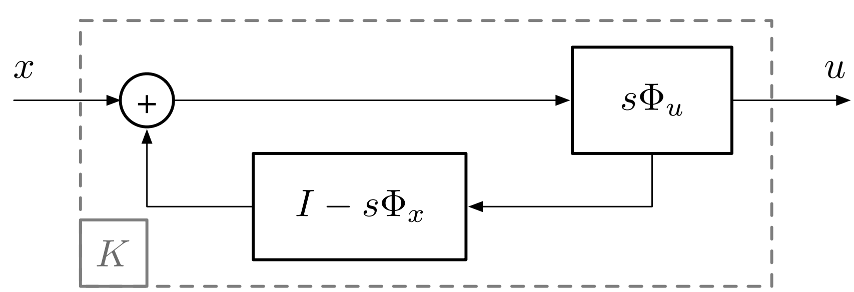


Figure 3.3: Implementation of state-feedback controller $u = Kx$ via the corresponding closed-loop mappings Φ^x and Φ^u. This implementation has a state space realization which inherits the structure of the closed-loops.

Note that if Φ_x and Φ_u satisfy (3.12), then $(I - s\Phi_x)$ is strictly proper, so that implementation (3.13) is well-defined. Just as in the more general output feedback setting, if $\Phi_x(s)$ and $\Phi_u(s)$ are TF-structured, then the corresponding controller is structured-realizable.

In this state-feedback setting, the structured-closed-loop, relative controller design problem (3.9) is written as

$$\left.\begin{array}{ll} \inf\limits_{K} & \left\| \mathcal{F}(P;K) = \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi_x(s) \\ \Phi_u(s) \end{bmatrix} B_1 \right\|_{\mathcal{H}_2} \\ \text{s.t.} & \Phi_x, \Phi_u \text{ are strictly proper} \\ & \Phi_x, \Phi_u \text{ are TF-structured w.r.t. } \mathcal{A}^{(b)} \qquad \text{(locality)} \\ & K\mathbb{1} = 0 \qquad \text{(relative feedback)} \end{array}\right\} \tag{3.14}$$

3.3.2 Consensus of 1$^{\text{st}}$ order subsystems

We next introduce the example that will be utilized to prove our main result. Throughout the remainder of this section and the next, the underlying graph $\mathcal{A}$ is a ring of N nodes with nearest neighbor edges, i.e. $\mathcal{A}_{ij} = 1$ when $|i-j| = 1$ with $|i-j|$ computed modulo N. We consider the problem of state-feedback design for distributed consensus of N 1$^{\text{st}}$-order subsystems over the undirected torus $\mathbb{Z}_N$:

$$\dot{x}_n(t) = u_n(t) + w_n(t), \;\; n \in \mathbb{Z}_N,$$

where x_n, u_n, w_n are the state, control action, and local disturbance at spatial location n, respectively. In vector form,

$$\begin{aligned} \dot{x} &= u + w, \\ z &= \begin{bmatrix} C \\ 0 \end{bmatrix} x + \begin{bmatrix} 0 \\ \gamma I \end{bmatrix} u, \end{aligned} \tag{3.15}$$

where z captures a measure of consensus, Cx, and a scaled measure of the control effort, γu.

Note that the open-loop system is not a consensus algorithm. Instead the controller K designed to optimize this performance output z will be such that the closed-loop system is a consensus algorithm.

We make the assumption that C is *circulant*, so that the open loop plant is spatially-invariant. We further restrict to relative C, i.e. $C\mathbb{1} = 0$; as a result, the closed-loop system may have a marginally stable mode at the origin representing the motion of the mean, which is undetectable through C. In particular, we allow for instability of

$$\mathcal{F}(\tilde{P};K) = \begin{bmatrix} \Phi_x(s) \\ \Phi_u(s) \end{bmatrix}$$

due to a pole at the origin of Φ_x, but stability of

$$\mathcal{F}(P;K) = \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \mathcal{F}(\tilde{P};K) I$$

is required for the objective of (3.14) to remain finite. This problem has been studied in e.g. [3,23,24]. The main result of this chapter is stated formally in the following theorem.

Theorem 3.3.2 *Let*

$$z = \mathcal{F}(P;K)w = \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} w$$

denote the closed-loop mapping for (3.15) *in feedback with a controller* $u = Kx$. *Assume that* C *is a circulant matrix with* $C\mathbb{1} = 0$ *with*

$$r = \text{rank } (C).$$

If $r > (2b+1)$, *then the TF-structured-closed-loop, relative controller design problem* (3.14) *is infeasible. In other words, any controller* K *which is relative and closed-loop TF-structured with respect to* $\mathcal{A}^{(b)}$ *will result in an unstable closed-loop* $\mathcal{F}(P;K)$.

Remark 5 *It is straightforward to show that if* K *is relative, then* K *will have a minimal* relative *realization, i.e. with "B" and "D" matrices relative. However, it is unclear whether a relative and closed-loop structured* K *will have a realization that is both structured and relative. Thus, we do not impose the additional restriction of a relative structured realization in* (3.14), *noting that optimization over this smaller subset would still result in infeasibility.*

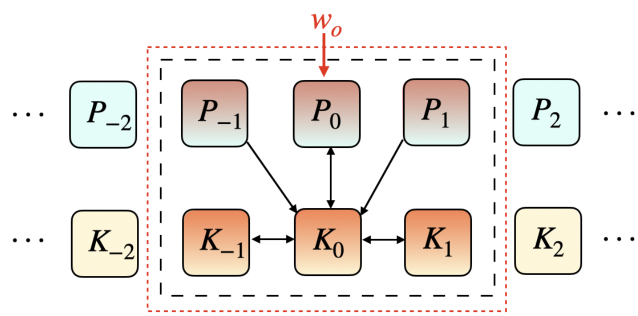
One interpretation of Theorem 3.3.2 is the following. It may be desirable to achieve consensus with a control policy implemented in a *local* manner, but a goal of consensus requires the corresponding closed-loop mappings to be full, e.g. every impulse response should be dense and *global* at least eventually. Thus, localization of closed-loop mappings and consensus present two opposite and conflicting design requirements. This result provides a step toward better understanding such conflicting measures.

Enforcing that the closed-loops are TF-structured *localizes* the propagation of disturbances, e.g. if $(\Phi_x)_{ij}(s) = (\Phi_u)_{ij}(s) = 0$ then a disturbance entering at spatial site j will not be seen by spatial site i for all time (see Figure 3.4a). This is a stringent requirement to employ to ensure structured-realizability of the controller, and we have just seen that it is far from being necessary in the context of relative feedback design problems. The difference between a structured-realizable controller and the stronger notion of a closed-loop TF-structured controller is depicted in Figure 3.4.

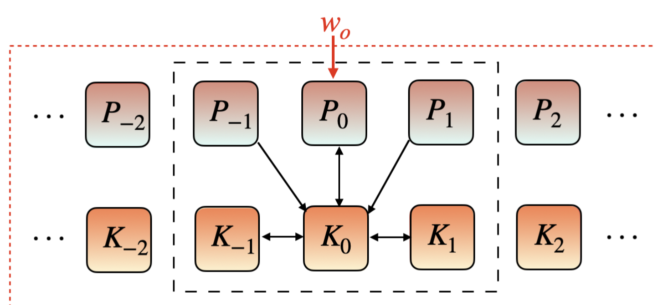
Corollary 3.3.3 *Let* $y = Cx$ *correspond to one of the following standard measures of consensus:*

- Local Error:

$$y_n = \left(C^{LE}x\right)_n := x_n - x_{n-1}.$$



(a) A plant P in feedback with closed-loop TF-structured controller for P. Effects of a disturbance are not seen outside a neighborhood of size one for all time. This neighborhood (specified by $\mathcal{A}$) is enclosed by the red dotted line.



(b) A plant P in feedback with a structured-realizable (but not closed-loop TF-structured) controller. Effects of a disturbance may eventually propagate to all spatial sites.

Figure 3.4: A comparison of plant P in feedback with: (*top*) a closed-loop TF-structured controller for P, and (*bottom*) a structured-realizable (but not closed-loop TF-structured) controller. A disturbance entering into the subplant P_0 at spatial site 0 is denoted w_0. In both cases, the controller is implementable using only local information; the black arrows denote the communication of subcontroller K_0, which is restricted to the neighborhood of size one illustrated with the black dashed line. A red gradient indicates that this disturbance may eventually affect a particular component; all potentially effected sites are enclosed by the red dotted line.

- Deviation from Average:

$$y_n = \left(C^{Ave}x\right)_n := x_n - \frac{1}{N}\sum_{i=0}^{N-1} x_i.$$

- Long Range Deviation:

$$y_n = \left(C^{LR}x\right)_n := x_n - x_{(n-N/2)}.$$

If K is relative and closed-loop TF-structured for (3.15) *with respect to $\mathcal{A}^{(b)}$ for any nontrivial choice of b ($1 \leq b < \frac{N}{2}$), then $\mathcal{F}(P;K)$ is unstable, i.e. problem* (3.14) *is infeasible.*

Proof: By Theorem 3.3.2, it is sufficient to show that $C^{\text{LE}}, C^{\text{Ave}}$, and C^{LR} all have rank $r = (N-1)$. This straightforward calculation is omitted. ■

This corollary demonstrates that this infeasibility occurs with both the *local* measure of consensus, C^{LE}, as well as for the *global* measures of consensus, C^{Ave} and C^{LR}.

3.3.3 Proof of Theorem 3.3.2

The key step in proving Theorem 3.3.2 is to write (3.14) in terms of the resulting closed-loops. We first utilize the following lemma to write the relative feedback constraint as a convex constraint on the closed-loops; a proof of this result is provided in Appendix 3.7.5.

Lemma 3.3.4 *Let P denote a plant of the form* (3.10) *with A relative and B full rank, and let $u = Kx$ be the state feedback control for this system. Then K is relative if and only if the corresponding closed-loop transfer function Φ_u is relative, i.e.*

$$K(s)\mathbb{1} = 0 \;\Leftrightarrow\; \Phi_u(s)\mathbb{1} = 0.$$

In particular, a state feedback controller K for (3.15) *is relative if and only if the corresponding map Φ^u is relative.*

Lemma 3.3.4 allows the structured-closed-loop, relative state-feedback controller design problem (3.14) to be written as:

$$\left.\begin{array}{ll}
\inf\limits_{\Phi_u,\Phi_x} & \left\| \mathcal{F}(P;K) = \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \\
\text{s.t.} & \begin{bmatrix} sI - 0 & -I \end{bmatrix} \begin{bmatrix} \Phi_x(s) \\ \Phi_u(s) \end{bmatrix} = I \\
& \Phi_x, \Phi_u \text{ strictly proper} \\
& \Phi_x, \Phi_u \text{ TF-structured w.r.t. } \mathcal{A}^{(b)} \qquad \text{(locality)} \\
& \Phi_u \mathbb{1} = 0 \qquad \text{(relative feedback)}
\end{array}\right\} \qquad (3.16)$$

Rearranging the affine subspace constraint (3.12) as

$$\Phi_x = \frac{1}{s}\left(I + \Phi_u\right), \tag{3.17}$$

we see that if Φ_u is strictly proper and TF-structured with respect to $\mathcal{A}^{(b)}$, then Φ_x will be strictly proper and TF-structured with respect to $\mathcal{A}^{(b)}$ as well. Then, (3.16) can be written as:

$$\left.\begin{array}{ll} \inf\limits_{\Phi_u} & \left\| \mathcal{F}(P;K) = \begin{bmatrix} C & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \frac{1}{s}\left(I - \Phi_u\right) \\ \Phi_u \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & \Phi_u \text{ strictly proper} \\ & \Phi_u \text{ TF-structured w.r.t. } \mathcal{A}^{(b)} \qquad \text{(locality)} \\ & \Phi_u \mathbb{1} = 0 \qquad \text{(relative feedback)} \end{array}\right\} \tag{3.18}$$

The techniques presented in [21], [22], can be used to convert (3.18) to an unconstrained model matching problem, which can be solved using standard techniques when the relative feedback assumption is removed. With this relative feedback constraint however, any Φ^u in the constraint set of (3.18) leads to an unstable $\mathcal{F}(P;K)$. The details of this argument are provided in Appendix 3.7.5.

3.4 SLS Performance Gap

The TF-structured-closed-loop, relative controller design problem (3.16) provides a convex relaxation of the structured-realizable, relative controller design problem. This convex relaxation was shown to be infeasible for the consensus example (3.15) presented in Section 3.3; a logical next question then is whether the structured-realizable, relative controller design problem is infeasible for this example as well. We study this question for the case of a deviation from average consensus metric. We show that for this same example, relative and structured controllers (without TF-structured closed loops) that achieve finite $\mathcal{H}_2$ norm are easily constructed. This shows that the performance gap between (3.6) and the convex relaxation provided by (3.9) may be infinite.

Consider the controller given by the static gain

$$K_s := -\begin{bmatrix} 2 & -1 & 0 & \cdots & -1 \\ -1 & 2 & -1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ -1 & 0 & 0 & \cdots & 2 \end{bmatrix}. \tag{3.19}$$

This controller was presented in e.g. [23], and the resulting closed-loop is a well-known nearest neighbor consensus algorithm. K_s is relative and an $\mathcal{A}$-structured realization is given by $\left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & K_s \end{array}\right]$. To see that K_s results in a finite closed-loop $\mathcal{H}_2$ norm, compute

$$\begin{aligned} \|\mathcal{F}(P;K_s)\|_{\mathcal{H}_2}^2 &= \operatorname{tr}\int_0^\infty e^{K^*t}\begin{bmatrix} C^{\text{Ave}} \\ \gamma K_s \end{bmatrix}^* \begin{bmatrix} C^{\text{Ave}} \\ \gamma K_s \end{bmatrix} e^{Kt}dt \\ &= (N+1)\gamma^2 + \frac{1}{4}\sum_{n=1}^{M-1}\frac{1}{1-\cos\left(\frac{2\pi n}{N}\right)} < \infty. \end{aligned} \tag{3.20}$$

This shows that for any choice of b, there exists a relative, $\mathcal{A}$-realizable controller that results in a stable $\mathcal{F}(P;K)$; this controller however is not closed-loop TF-structured. Thus, the performance gap between the optimal relative, structured-realizable controller and the optimal relative, closed-loop TF-structured controller is *infinite* in this setting.

A potential drawback of the controller K_s is that it is not network-realizable; in particular, its implementation requires instantaneous access to measurements of neighboring subsystems through the tridiagonal "D" matrix. The following example provides a strictly proper approximation of K_s, which eliminates this need for instant information from neighboring subsystems.

Example 3.4.1 *Let K_a denote a strictly proper approximation of K_s defined by the realization*

$$K_a := \left[\begin{array}{c|c} aI & K_s \\ \hline -aI & 0 \end{array}\right],$$

for $a < 0$. Because the "C" and "D" matrices of this realization are non-zero only on the diagonal, only instant access to a subcontrollers own local information is required. K_a is relative, and network-realizable. Note that K_a is TF-structured with respect to $\mathcal{A}$ as well. As $a \to -\infty$, the closed-loop performance achieved by K_a converges in $\mathcal{H}_2$ norm to the performance achieved by K_s.

As K_s and K_a achieve finite closed-loop $\mathcal{H}_2$ cost, Theorem 3.3.2 implies that neither of these controllers are closed-loop TF-structured. This can also be seen directly by noting that the corresponding closed-loop mappings $\Phi_x = (sI - 0 - I \cdot K_s)^{-1}$ and $\Phi_x = (sI - 0 - I \cdot K_a(s))^{-1}$ are full transfer function matrices. This proves the following additional relationships among the different notions of structure.

Lemma 3.4.1 *The converse of Theorem 3.2.3 does not hold: there exist structured-realizable controllers which are not closed-loop TF-structured. In addition, there exist TF-structured controllers which are not closed-loop TF-structured.*

The relations between the different notions of structure are depicted in Figure 3.5 and are summarized in the following diagram.

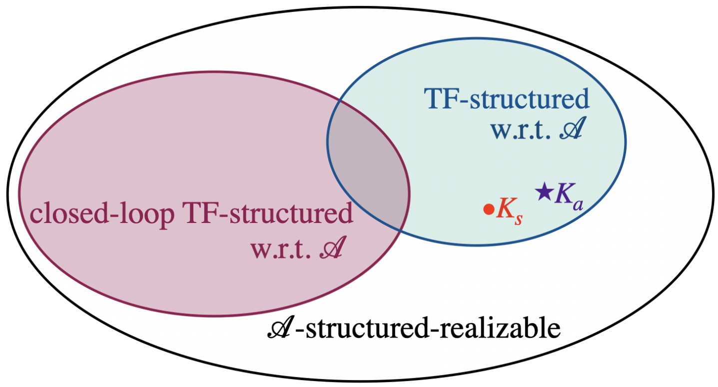
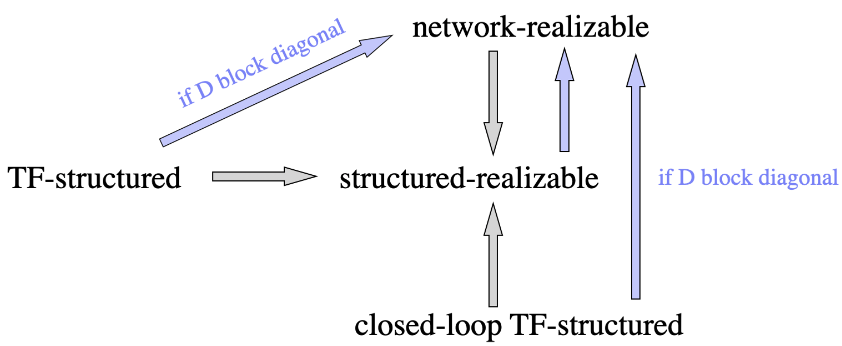


Figure 3.5: A Venn diagram depicting the set relations between closed-loop TF-structure, TF-structure, and structured-realizability. The sets of closed-loop TF-structured controllers and TF-structured controllers are both subsets of the set of structured controllers; in general this subset relation is strict. The sets of closed-loop TF-structured and TF-structured controllers are in general non-comparable; their intersection is non-empty and neither is a strict subset of the other. The static, TF-structured controller K_s (3.19) is depicted by the star, and its strictly proper approximation K_a is denoted by the dot; neither of these controllers is closed-loop structured for the plant P of interest.



Note that in general the sets of closed-loop TF-structured controllers and TF-structured controllers are incomparable: their intersection is non-empty, and neither is a strict subset of the other.

3.5 Extension to Higher Spatial Dimensions

It has been shown [23] that static relative controllers with subcontroller communication limited to within a prescribed distance are able to regulate large-scale disturbances for the consensus problem in spatial dimension $d = 3$ but not in spatial dimension $d = 1$, i.e. such control policies scale poorly with network size for the consensus of subsystems on the undirected torus $\mathbb{Z}_N$ but not on the undirected 3−dimensional torus $\mathbb{Z}_N^3$. A similar result for controllers with locally 1$^{\text{st}}$-order dynamics was demonstrated in [24]. Theorem 3.3.2 demonstrated a limitation of relative and closed-loop structured controllers for the consensus problem in the spatial dimension $d = 1$. Based on the findings of e.g. [23, 24], a relevant question is whether the results of this theorem carry over to the higher spatial dimension setting. To answer this question, we first present some preliminaries on system dynamics in higher spatial dimensions. For simplicity of exposition, we restrict our analysis to the *spatially-invariant* setting.

Given an array $a \in \mathbb{Z}_N^d$, we denote by T_a the operator of circular convolution with the array a:

$$(T_a x)_n = \sum_{m \in \mathbb{Z}_N^d} a_m x_{n-m},$$

using multi-index notation, e.g. $a_n = a_{(n_1,\ldots,n_d)}$.

Example 3.5.1 *We let $\mathbb{1} \in \mathbb{Z}_N^d$ denote the array of all ones and let $\delta \in \mathbb{Z}_N^d$ denote the array defined by*

$$\delta_n = \begin{cases} 1, & n = 0 \\ 0, & n \in \mathbb{Z}_N^d \setminus \{0\}. \end{cases}$$

Then T_δ is the identity operator, i.e. $(Ix)_n = (T_\delta x)_n := x_n$, and $T_{\mathbb{1}}$ is defined by

$$(T_{\mathbb{1}} x)_n := \sum_{m \in \mathbb{Z}_N^d} x_m. \tag{3.21}$$

We consider systems with *spatially-invariant* dynamics:

$$\begin{aligned} \dot{x}(t) &= (T_a x)(t) + (T_{b_1} w)(t) + (T_{b_2} u)(t) \\ z(t) &= (T_{c_1} x)(t) + (T_{d_{12}} u)(t), \end{aligned} \tag{3.22}$$

where $x, u, w,$ and z are vector-valued functions on the undirected torus $\mathbb{Z}_N^d$. $x, u, w,$ and z represent the spatially distributed state, control signal, exogenous disturbance and performance output, respectively so that e.g. $x_n(t)$ represents the state at spatial site $n \in \mathbb{Z}_N^d$. We consider the design of a state-feedback controller K which also has a spatially-invariant representation, i.e. is a *spatially-invariant system.*

Definition 3.5.1 *An LTI mapping H from signal $x \in \mathbb{Z}_N^d$ to signal $u \in \mathbb{Z}_N^d$ is a* spatially-invariant system *if operation by H can be written as spatial convolution in the transfer function domain, i.e.*

$$\begin{aligned} (HX)_n(s) &= (T_h X)_n(s) \\ &:= \sum_{m \in \mathbb{Z}_N^d} h_m(s) X_{n-m}(s) = H(s)X(s). \end{aligned} \tag{3.23}$$

H is completely specified by the sequence of transfer functions $\{h_m(s)\}_{m \in \mathbb{Z}_N^d}$, which we refer to as its convolution kernel. *H is said to be* stable *if for each m, h_m of the convolution kernel is stable, and* strictly proper *if each h_m is strictly proper. The $\mathcal{H}_2$ norm of H is given by*

$$\|H\|_{\mathcal{H}_2}^2 := \sum_{m \in \mathbb{Z}_N^d} \|h_m\|_{\mathcal{H}_2}^2,$$

where we have normalized by the number of subsystems N^d and exploited the spatial invariance property.

The closed-loop mappings from w to x and u for system (3.22) in feedback with a spatially-invariant state-feedback controller K are given by

$$\begin{aligned} X &= (sI - T_a - T_{b_2}K)^{-1}T_{b_1}W &&=: \Phi_x T_{b_1} W, \\ U &= K(sI - T_a - T_{b_2}K)^{-1}T_{b_1}W &&=: \Phi_u T_{b_1} W. \end{aligned} \tag{3.24}$$

Φ_x and Φ_u defined by (3.24) are themselves spatially-invariant systems [20], specified by the convolution kernels $\{(\phi_x)_m(s)\}_{m\in\mathbb{Z}_N^d}$ and $\{(\phi_u)_m(s)\}_{m\in\mathbb{Z}_N^d}$ respectively.

Extending the example considered in spatial dimension one, we consider the problem of distributed consensus of N^d 1st-order subsystems on the undirected d-dimensional torus $\mathbb{Z}_N^d$.

We design a controller $u = Kx$ for (3.26) that is relative and spatially-invariant, measuring performance with the $\mathcal{H}_2$ norm of the spatially-invariant closed-loop mapping $\mathcal{F}(P;K)$. We restrict to controllers that are *closed-loop TF-structured*, extending this notion to the higher-spatial-dimensional setting. We let $\mathcal{A}$ be a generalization of the graph defined in Section 3.3; thus $\mathcal{A}$ refers to the d-dimensional torus with N^d nodes with nearest neighbor edges, i.e. there exists an edge between node $i = (i_1, ..., i_d)$ and node $j = (j_1, ..., j_d)$ if $|i_k - j_k| \le 1$ for all $k = 1, ..., d$ with $|i_k - j_k|$ computed modulo N. We similarly let $\mathcal{A}^{(b)}$ denote the corresponding b-hops graph.

Definition 3.5.2 *A spatially-invariant controller K is* closed-loop TF-structured *for plant P with respect to $\mathcal{A}^{(b)}$ if the convolution kernels of the resulting closed-loops Φ_x and Φ_u are* TF-structured *with respect to $\mathcal{A}^{(b)}$, i.e. the convolution kernels $\{(\phi_u)_n(s)\}_{n\in\mathbb{Z}_N^d}$ and $\{(\phi_u x_n(s)\}_{n\in\mathbb{Z}_N^d}$ satisfy $(\phi_u)_n(s) = (\phi_x)_n(s) = 0$ whenever $\max_{1\le j\le d}|n_j| > b$.*

We additionally restrict to relative feedback controllers; this requirement is compactly characterized as follows.

Lemma 3.5.1 *A spatially-invariant system K is relative if and only if $KT_{\mathbb{1}} = 0$.*

The relative, closed-loop TF-structured design problem in this higher dimensional setting can be formally stated as:

$$\left.\begin{array}{lll} \inf\limits_{K} & \|\mathcal{F}(P;K)\|_{H_2}^2 & \\ \text{s.t.} & K \text{ spatially-invariant}, & \\ & K \text{ closed-loop TF-structured w.r.t. } \mathcal{A}^{(b)} & \text{(locality)} \\ & KT_{\mathbb{1}} = 0 & \text{(relative feedback)} \end{array}\right\} \tag{3.25}$$

We consider an extension of the example provided in Section 3.4 to the higher dimensional setting. The plant dynamics are given in the form (3.22) by

$$\begin{aligned} \dot{x} &= u + w, \\ z &= \begin{bmatrix} T_c \\ 0 \end{bmatrix} x \; + \; \begin{bmatrix} 0 \\ \gamma I \end{bmatrix} u, \end{aligned} \tag{3.26}$$

where $T_c := I - \frac{1}{N^d} T_{\mathbb{1}}$, i.e. $(T_c x)_n = x_n - \frac{1}{N^d} \sum_{m \in \mathbb{Z}_N^d} x_m$, so that the output z captures a deviation from average measure of consensus, $T_c x$, and a weighting of the control action, γu. The closed-loop from disturbance to performance output is given by

$$z = \mathcal{F}(P;K)w = \begin{bmatrix} T_c & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} w.$$

Theorem 3.5.2 *Problem* (3.25) *is infeasible: if $u = Kx$ is relative, spatially-invariant, and closed-loop TF-structured for* (3.26) *with respect to $\mathcal{A}^{(b)}$ for any $1 \leq b < \frac{N}{2}$, then $\mathcal{F}(P;K)$ is not stable.*

The key idea of this proof is to use a generalizations of Lemmas 3.3.1 and 3.3.4 to write (3.25) as

$$\left.\begin{array}{lll} \inf\limits_{\Phi_x, \Phi_u} & \left\| \mathcal{F}(P;K) = \begin{bmatrix} T_c & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \Phi_x \\ \Phi_u \end{bmatrix} \right\|_{\mathcal{H}_2}^2 & \\ \text{s.t.} & s\Phi_x - \Phi_u = I, & \\ & \Phi_u, \Phi_x \text{ strictly proper}, & \\ & \Phi_u, \Phi_x \text{ TF-structured w.r.t. } \mathcal{A}^{(b)} & \text{(locality)} \\ & \Phi_u T_{\mathbb{1}} = 0 & \text{(relative feedback)} \end{array}\right\} \quad (3.27)$$

Note that (3.27) simplifies to (3.16) (with an additional spatial invariance constraint) in the case of dimension $d = 1$. The details of this reformulation, along with a proof that (3.27) is infeasible are provided in Appendix 3.7.7.

3.6 Discussion & Open Problems

Optimal design of a distributed controller that is structured (or network) realizable with respect to an underlying communication graph remains an open problem in general settings. The SLS approach provides an upper bound by optimizing over the set of *closed-loop TF-structured* controllers; this allows for a search over dynamic controllers with arbitrary amounts of memory and is tractably solvable via a convex optimization problem. In this chapter, we demonstrated that this closed-loop structured optimal controller design problem may be infeasible, when relative feedback constraints are imposed. This highlights that imposing sparsity structure on the closed-loop maps is perhaps too heavy-handed of a restriction to ensure structured implementation. The SLS framework however does enable the imposition of other structural constraints on the closed loop, provided they are convex. There maybe other (than specified sparsity) closed-loop structures that still enable the implementation of structured-realizable controllers from a closed loop design. This is the subject of current and future research. We point out that recent works such as [34] have also begun to address this disconnect between controller structure and closed-loop structure outside of special problem settings.

The question of how to *realize* a controller in the distributed setting remains an open problem. In this setting, it is no longer desirable to implement controllers using minimal state-space realizations; instead some sort of tradeoff between local controller state dimension and subcontroller communication requirements should be considered. SLS has begun to address this question [11], and recent works including [9,10,12–14,32,35] have added to this discussion.

3.7 Appendix

3.7.1 Proof of Lemma 3.2.1

Let G be TF-structured with respect to $\mathcal{A}$. We construct a structured realization of G as follows. For all $G_{ij} \neq 0$, define $A_{ij}, B_{ij}, C_{ij}, D_{ij}$ from any realization $G_{ij} = C_{ij}(sI - A_{ij})^{-1}B_{ij} + D_{ij}$, and for $G_{ij} = 0$, define A_{ij}, B_{ij}, and C_{ij} to be empty matrices, and $D_{ij} = 0$.

We can realize each row G_i of G as

$$G_i = \left[\begin{array}{c|c} \overline{A}_i & \overline{B}_i \\ \hline \overline{C}_i & \overline{D}_i \end{array}\right] := \left[\begin{array}{ccc|ccc} A_{i1} & & & B_{i1} & & \\ & \ddots & & & \ddots & \\ & & A_{i,N} & & & B_{i,N} \\ \hline C_{1i} & \cdots & C_{1,N} & D_{1i} & \cdots & D_{1,N} \end{array}\right].$$

The entire system G can then be realized as

$$G = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right] := \left[\begin{array}{ccc|ccc} \overline{A}_1 & & & & \overline{B}_1 & \\ & \ddots & & & \vdots & \\ & & \overline{A}_N & & \overline{B}_N & \\ \hline \overline{C}_1 & \cdots & \overline{C}_N & \overline{D}_1 & \cdots & \overline{D}_N \end{array}\right]$$

$$= \left[\begin{array}{ccc:c:ccc|c:c:c} A_{1,1} & & & & & & & B_{1,1} & & \\ & \ddots & & & & & & & \ddots & \\ & & A_{1,N} & & & & & & & B_{1,N} \\ \hdashline & & & \ddots & & & & & \vdots & \\ \hdashline & & & & A_{N,1} & & & B_{N,1} & & \\ & & & & & \ddots & & & \ddots & \\ & & & & & & A_{N,N} & & & B_{N,N} \\ \hline C_{1,1} & \cdots & C_{1,N} & & & & & D_{1,1} & \cdots & D_{1,N} \\ \hdashline & & & \ddots & & & & \vdots & & \vdots \\ \hdashline & & & & C_{N,1} & \cdots & C_{N\text{-}1,N} & D_{N,1} & \cdots & D_{N,N} \end{array}\right],$$

where the dashed lines represent the partitioning of inputs, outputs and states according to site index. It is clear that A and C are block diagonal, and therefore $A, C \in \mathcal{S}(\mathcal{A})$

trivially. The matrices B and D have a block structure such that the ij'th block is zero if $\mathcal{A}_{ij} = 0$, i.e. $B, D \in \mathcal{S}(\mathcal{A})$. Thus the realization (A, B, C, D) is $\mathcal{A}$-structured.

Note that if the D matrix is block diagonal, then both C and D are block diagonal, and the above system is thus also network-realizable. Note that a similar construction to the above can alternatively yield a block-diagonal B matrix if we begin by realizing each column of G rather than each row.

The converse does not hold, as demonstrated by the following example. Let

$$\mathcal{A} := \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

and define $G = \left[\begin{array}{c|c} A & I \\ \hline I & 0 \end{array}\right]$, with $A := \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$. Then G is structured-realizable (also network-realizable), but a direct calculation shows that $G(s) = I(sI - A)^{-1}I$ is not TF-structured with respect to $\mathcal{A}$.

3.7.2 Relative Measurements and the proof of Lemma 3.2.2

Denote the n'th row of K by K^n. It is clear that the condition $K\mathbb{1} = 0$ applies row-by-row, i.e.

$$K\,\mathbb{1} = 0 \quad \Leftrightarrow \quad K^n\,\mathbb{1} = 0,\ n = 1, \ldots, N,$$

and we therefore can examine the relation between each scalar output and all inputs individually and drop the superscript n.

One direction in the proof of Lemma 3.2.2 is easy. If K is relative, meaning that it can be written in the form (3.5), then if we act on the vector $\mathbb{1}$ with it

$$K(s)\mathbb{1} \;=\; \sum_{1 \le i < j \le N} \mathcal{K}_{ij}(s)\ (1 - 1) \;=\; 0.$$

For simplicity of notation, we will drop the argument s from transfer functions going forward.

The other direction is to show that if $K\mathbb{1} = 0$, then we can rewrite Ky in the form (3.5)

$$u \;=\; Ky \;=\; \sum_{1 \le i < j \le N} \mathcal{K}_{ij}\ (y_i - y_j). \tag{3.28}$$

Without loss of generality, we can assume each $\mathcal{K}_{ij}$ to be SISO, which is equivalent to assuming each signal y_i to be scalar. If Ky can be written in the above form for each of the scalar subcomponent of each y_i, then concatenating these representations as columns would give the representation for vector signals $\{y_i\}$.

Note that the form (3.28) involves $(N^2 - N)/2$ transfer functions $\{\mathcal{K}_{ij}\}$. It is useful to rewrite this relation using more compact matrix notation. Form the $N \times N$ skew-symmetric (not skew-Hermitian) transfer function matrix

$$\mathcal{K} := \begin{bmatrix} 0 & -\mathcal{K}_{12} & \cdots & -\mathcal{K}_{1N} \\ \mathcal{K}_{12} & 0 & & \\ \vdots & & \ddots & \\ \mathcal{K}_{1N} & & & 0 \end{bmatrix}, \tag{3.29}$$

Then the relation (3.28) can be compactly written

$$Ky = \mathbb{1}^T \mathcal{K} y.$$

Therefore the relation between K and $\mathcal{K}$ (after transposing) is

$$\begin{bmatrix} 0 & & \mathcal{K}_{ij} \\ & \ddots & \\ -\mathcal{K}_{ij} & & 0 \end{bmatrix} \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \begin{bmatrix} K_1 \\ \vdots \\ K_N \end{bmatrix}. \tag{3.30}$$

If we think about solving for $\mathcal{K}$ from a given K using the equation above, it is clearly a highly underdetermined system of linear equations. If one solution exists, then there are an infinite number of other solutions.

We want to characterize when there exits solutions to (3.30). We can actually say more than that. It is possible to characterize when there exists solutions with a particular sparsity structure, i.e. where K is rewritten in (3.28) using only differences $y_i - y_j$ with the pairings (i, j) selected as the edges of a pre-specified graph. We state this formally.

Lemma 3.7.1 *Let $\mathcal{A}$ be the adjacency matrix of an undirected graph. Let K in (3.30) be such that $K\mathbb{1} = 0$, then there exists a solution $\{\mathcal{K}_{ij}\}$ to (3.30) with the same sparsity structure as $\mathcal{A}$, i.e.*

$$\mathcal{A}_{ij} = 0 \quad \Rightarrow \quad \mathcal{K}_{ij} = 0$$

iff the graph $\mathcal{A}$ is connected.

Note that the graph $\mathcal{A}$ in this lemma is unrelated to the graph with which locality of transfer function matrices or their realizations is specified. The concepts of relative systems and structured systems are independent concepts. *Proof:*

Define the following sets of complex matrices

$$\mathcal{S} := \left\{ M \in \mathbb{C}^{N \times N};\ M^T = -M \right\}, \text{ (skew-symmetric matrices)}$$
$$\mathcal{S}_{\mathcal{A}} := \{ M \in \mathcal{S};\ \mathfrak{sp}(M) = \mathcal{A} \}, \text{ (skew-symmetric w/ sparsity } \mathcal{A}),$$

where $\mathfrak{sp}(M)$ stands for the matrix of the sparsity pattern of M. Note that these sets are vector spaces. Now consider the matrix operator

$$\mathcal{L}_{\mathcal{S}_{\mathcal{A}}} : \mathcal{S}_{\mathcal{A}} \to \mathbb{C}^N, \qquad \mathcal{L}_{\mathcal{S}_{\mathcal{A}}}(M) := M\mathbb{1}.$$

The solvability of

$$\mathcal{K}^T(s)\, \mathbb{1} \;=\; K^T(s), \tag{3.31}$$

with a solution of same sparsity as $\mathcal{A}$ is equivalent to the solvability of

$$\mathcal{L}_{\mathcal{S}_\mathcal{A}}\left(\mathcal{K}(s)\right) \;=\; K^T(s),$$

which in turn is equivalent to the statement

$$K^T(s) \;\in\; \mathrm{Im}(\mathcal{L}_{\mathcal{S}_\mathcal{A}})\,, \tag{3.32}$$

where Im(.) denotes the image space of the operator. All of the above statements are to be interpreted as required to hold for each $s \in \mathbb{C}$ except at isolated points.

Thus we have converted the solvability question to one about the range space of a matrix operator. Recall the following important consequence of the fundamental theorem of linear algebra

$$\mathrm{Im}(\mathcal{L}_{\mathcal{S}_\mathcal{A}}) \;=\; \mathrm{Im}\Big(\mathcal{L}_{\mathcal{S}_\mathcal{A}}\mathcal{L}^{\dagger}_{\mathcal{S}_\mathcal{A}}\Big)\,,$$

where $\mathcal{L}^{\dagger}_{\mathcal{S}_\mathcal{A}} : \mathbb{C}^N \rightarrow \mathbb{C}^{N\times N}$ is the adjoint. It turns out that it is much easier to characterize $\mathcal{L}_{\mathcal{S}_\mathcal{A}}\mathcal{L}^{\dagger}_{\mathcal{S}_\mathcal{A}}$ in terms of the graph connectivity as stated in the following lemma whose proof is given below.

Lemma 3.7.2 *The composition of $\mathcal{L}_{\mathcal{S}_\mathcal{A}}$ with its adjoint is given by*

$$\mathcal{L}_{\mathcal{S}_\mathcal{A}}\mathcal{L}^{\dagger}_{\mathcal{S}_\mathcal{A}} \;=\; \frac{1}{2}\, L$$

where L is the Laplacian of the graph $\mathcal{A}$.

For undirected networks, L is a symmetric matrix, and thus its image and null spaces are mutually orthogonal. A standard result in algebraic graph theory states that a graph is connected iff the null space of L is just $\mathbb{1}$, i.e. it is connected iff

$$\mathrm{Nu}(L) = \mathrm{span}(\mathbb{1}) \qquad \Leftrightarrow \qquad \mathbb{1}^{\perp} = \mathrm{Im}(L)\,.$$

Note that the condition (3.32) is required not for any $K(s)$, but only those that are such that $K(s)\mathbb{1} = 0$, i.e. $K(s) \in \mathbb{1}^{\perp}$, which is exactly $\mathrm{Im}(L)$. We therefore conclude that

$$K(s)\mathbb{1} = 0 \quad \Rightarrow \quad K^T(s) \in \mathrm{Im}(L) = \mathrm{Im}\Big(\mathcal{L}_\mathcal{A}\mathcal{L}^{\dagger}_\mathcal{A}\Big) = \mathrm{Im}(\mathcal{L}_\mathcal{A})\,,$$

and the equation (3.31) is solvable with a $\mathcal{K}(s)$ that has the same sparsity structure as the graph $\mathcal{A}$.

■

Proof of Lemma 3.7.2

The first step is to compute the adjoint $\mathcal{L}_{\mathcal{S}_\mathcal{A}}^\dagger$, and then compute the composition $\mathcal{L}_{\mathcal{S}_\mathcal{A}}\mathcal{L}_{\mathcal{S}_\mathcal{A}}^\dagger$. To compute the adjoint, it is easier to work with the following operator

$$\mathcal{L} : \mathbb{C}^{N\times N} \to \mathbb{C}^N, \qquad \mathcal{L}(M) := M\mathbb{1},$$

and note that $\mathcal{L}_{\mathcal{S}_\mathcal{A}} = \mathcal{L}|_{\mathcal{S}_\mathcal{A}}$, i.e. the restriction of $\mathcal{L}$ to $\mathcal{S}_\mathcal{A}$. It then follows that

$$\mathcal{L}_{\mathcal{S}_\mathcal{A}}^\dagger = \Pi_{\mathcal{S}_\mathcal{A}}\mathcal{L}^\dagger = \Pi_{\mathcal{S}_\mathcal{A}}\Pi_{\mathcal{S}}\mathcal{L}^\dagger,$$

where we have written the projection as the composition of two projections that are each easier to compute. In summary, we have

$$\mathcal{S}_\mathcal{A} \xleftarrow{\Pi_{\mathcal{S}_\mathcal{A}}} \mathcal{S} \xleftarrow{\Pi_{\mathcal{S}}} \mathbb{C}^{N\times N} \xleftarrow{\mathcal{L}^\dagger} \mathbb{C}^N$$

Each of those three operators are easy to compute. If M is a skew-symmetric matrix, then

$$\Pi_{\mathcal{S}_\mathcal{A}} : \mathcal{S} \to \mathcal{S}_\mathcal{A}, \qquad \Pi_{\mathcal{S}_\mathcal{A}}(M) = \mathcal{A} \circ M,$$

where $\circ$ is the Hadamard (element-by-element) product of two matrices. Now if M is any complex matrix, then

$$\Pi_{\mathcal{S}} : \mathbb{C}^{N\times N} \to \mathcal{S}, \qquad \Pi_{\mathcal{S}}(M) = \frac{1}{2}\left(M - M^T\right).$$

Finally, given any complex vector v

$$\mathcal{L}^\dagger : \mathbb{C}^N \to \mathbb{C}^{N\times N}, \qquad \mathcal{L}^\dagger(v) = v\mathbb{1}^T.$$

The last fact follows from the requirement $\forall M \in \mathbb{C}^{N\times N}$

$$\operatorname{tr}\left(\left(\mathcal{L}^\dagger v\right)^* M\right) = \left\langle \mathcal{L}^\dagger v, M\right\rangle \equiv \langle v, \mathcal{L}(M)\rangle = \operatorname{tr}\left(v^* M\mathbb{1}\right).$$

Putting it all together, we conclude that

$$\mathcal{L}_{\mathcal{S}_\mathcal{A}}^\dagger v = \Pi_{\mathcal{S}_\mathcal{A}}\Pi_{\mathcal{S}}\mathcal{L}^\dagger v = \frac{1}{2}\,\mathcal{A} \circ \left(v\mathbb{1}^T - \mathbb{1}v^T\right)$$

Finally, we compute the composition $\mathcal{L}_{\mathcal{S}_\mathcal{A}}\mathcal{L}_{\mathcal{S}_\mathcal{A}}^\dagger$. First note that if $\mathcal{A}$ is the adjacency matrix of an undirected graph, the following is a useful characterization of the Laplacian

$$L = D - \mathcal{A} = \mathsf{diag}(\mathcal{A}\mathbb{1}) - \mathcal{A},$$

where D is the diagonal matrix of node degrees, which is $\mathsf{diag}(\mathcal{A}\mathbb{1})$, where $\mathsf{diag}(w)$ makes a diagonal matrix from the entries of the vector w. Now compute

$$\begin{aligned}
2\,\mathcal{L}_{\mathcal{S}_\mathcal{A}}\mathcal{L}^{\dagger}_{\mathcal{S}_\mathcal{A}} v &= \left[\mathcal{A}\circ(v\mathbb{1}^T - \mathbb{1}v^T)\right]\mathbb{1} \\
&= \left[\mathcal{A}\circ(v\mathbb{1}^T) - \mathcal{A}\circ(\mathbb{1}v^T)\right]\mathbb{1} \\
&\overset{1}{=} \left[\mathsf{diag}(v)\,\mathcal{A}\,\mathsf{diag}(\mathbb{1}) - \mathsf{diag}(\mathbb{1})\,\mathcal{A}\,\mathsf{diag}(v)\right]\mathbb{1} \\
&= \left[\mathsf{diag}(v)\,\mathcal{A}\,I - I\mathcal{A}\,\mathsf{diag}(v)\right]\mathbb{1} \\
&= \mathsf{diag}(v)\,\mathcal{A}\,\mathbb{1} - \mathcal{A}\,\mathsf{diag}(v)\,\mathbb{1} \\
&\overset{2}{=} \mathsf{diag}(\mathcal{A}\mathbb{1})\,v - \mathcal{A}\,v \;=\; Dv - \mathcal{A}v \;=\; Lv,
\end{aligned}$$

where $\overset{1}{=}$ follows from the fact that the Hadamard product with a rank one matrix $\mathcal{A}\circ(uw^T) = \mathsf{diag}(u)\,\mathcal{A}\,\mathsf{diag}(w)$, and $\overset{2}{=}$ follows from $\mathsf{diag}(v)\,w = \mathsf{diag}(w)\,v$ for any two vectors v and w.

3.7.3 Proof of Theorem 3.2.3

We first demonstrate that the inner feedback loop (denoted by $\frac{1}{s^2}(\Phi_{xx})^{-1}$ in Figure 3.1b has a structured realization with a zero "D" block (strictly proper). Since Φ_{xx} and Φ_{ux} are TF-structured w.r.t. $\mathcal{A}$ and strictly proper, we may construct realizations using Lemma 3.2.1:

$$\Phi_{xx} = \left[\begin{array}{c|c} A_x & B_x \\ \hline C_x & 0 \end{array}\right],$$

with $B_x \in \mathcal{S}(A)$ and A_x, C_x block diagonal. From (3.8), we see that $s\Phi_{xx} - (A\Phi_{xx} - B\Phi_{ux}) = I$; as $A\Phi_{xx}$ and $B\Phi_{ux}$ are strictly proper, the "D" block of a realization of $s\Phi_{xx}$ must be the identity:

$$s\Phi_{xx} = \left[\begin{array}{c|c} A_x & B_x \\ \hline C_xA_x & I \end{array}\right].$$

A realization of the inner feedback loop $\frac{1}{s^2}(\Phi_{xx})^{-1}$ in Figure 3.1b is then

$$\frac{1}{s^2}(\Phi_{xx})^{-1} = \left[\begin{array}{cc|c} A_x & B_x & 0 \\ -C_xA_x^2 & -C_xA_xB_x & I \\ \hline 0 & I & 0 \end{array}\right].$$

A_x and $-C_xA_x^2$ are block diagonal and $B_x, -C_xA_xB_x \in \mathcal{S}(\mathcal{A})$. Reordering the states appropriately leads to a realization of $\frac{1}{s^2}(\Phi_{xx})^{-1}$ that is structured w.r.t. $\mathcal{A}$:

$$\frac{1}{s^2}(\Phi_{xx})^{-1} = \left[\begin{array}{c|c} A_L & B_L \\ \hline C_L & 0 \end{array}\right],$$

$A_L \in \mathcal{S}(\mathcal{A})$, B_L, C_L block diagonal. The controller implementation is then depicted in terms of realizations in Figure 3.6, where realizations of the closed-loops are:

$$\Phi_{xy} = \left[\begin{array}{c|c} A_{xy} & B_{xy} \\ \hline C_{xy} & 0 \end{array}\right], \quad \Phi_{ux} = \left[\begin{array}{c|c} A_{ux} & B_{ux} \\ \hline C_{ux} & 0 \end{array}\right],$$

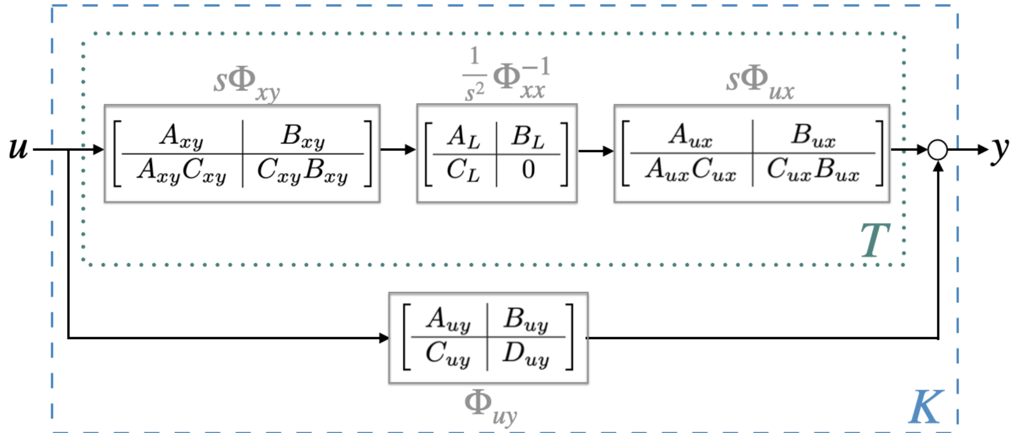


Figure 3.6: An implementation of the controller K resulting from the SLS design procedure, utilizing structured realizations of the corresponding closed-loop mappings. This implementation provides a state-space realization of K which is structured.

$$\Phi_{uy} = \left[\begin{array}{c|c} A_{uy} & B_{uy} \\ \hline C_{uy} & D_{uy} \end{array}\right],$$

with $A_{xy}, A_{ux}, A_{uy}, C_{xy}, C_{ux}, C_{xy}$ block diagonal and $B_{xy}, B_{ux}, B_{uy}, D_{xy} \in \mathcal{S}(\mathcal{A})$. The top cascade interconnection T depicted in Figure 3.6 is strictly proper, and each component of this interconnection is a structured realization w.r.t. $\mathcal{A}$. Moreover, each component of this cascade connection has a "C" matrix which is block diagonal. It can be shown, using arguments similar to that of [14] that (T) has a realization

$$T = \left[\begin{array}{c|c} A_T & B_T \\ \hline C_T & 0 \end{array}\right],$$

$A_T, B_T, C_T \in \mathcal{S}(\mathcal{A})$. Similarly, it can be shown that the parallel interconnection of T with Φ_{uy} (forming K) will be $\mathcal{A}$-structured-realizable.

3.7.4 Example for the non-equivalence of TF-structured and closed-loop TF-structured designs

We prove that closed-loop TF-structure does not imply TF-structure through an explicit counterexample. Let

$$\mathcal{A} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix},$$

P be a plant with dynamics $\dot{x} = u + w$, and the control $u = Kx$ for this plant be given by

$$K(s) = \frac{s}{(s+1)^2(s+2)^2 - (s+1)^2 - (s+2)^2} \cdot \begin{bmatrix} (s+2)^2 & (s+1)(s+3)^2 & -(s+1)(s+3) \\ (s+1)(s+3)^2 & -(s+1)^2-(s+3)^2 & (s+1)^2(s+3) \\ -(s+1)(s+3) & (s+1)^2(s+3) & -(s+1)^2 \end{bmatrix} \quad (3.33)$$

Then K is not TF-structured w.r.t. $\mathcal{A}$ since $K_{31}(s)$ and $K_{13}(s)$ are nonzero. The resulting closed-loop maps are given by

$$\Phi^x(s) = \frac{1}{s} \begin{bmatrix} 1 & \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & 1 & \frac{1}{s+2} \\ 0 & \frac{1}{s+2} & 1 \end{bmatrix},$$

$$\Phi^u(s) = \begin{bmatrix} 0 & \frac{1}{s+1} & 0 \\ \frac{1}{s+1} & 0 & \frac{1}{s+2} \\ 0 & \frac{1}{s+2} & 0 \end{bmatrix},$$

so that K is closed-loop TF-structured for plant P w.r.t. $\mathcal{A}$.

3.7.5 Proof of Lemma 3.3.4

Given a controller $u = Kx$, the resulting closed-loop transfer function Φ_u for system (3.15) is given by

$$\Phi_u(s) = K(s)(sI - A - BK(s))^{-1}.$$

First assume K is relative, i.e. $K(s)\mathbb{1} = 0$. Then

$$(sI - A - BK(s))\mathbb{1} = s\mathbb{1},$$

so that

$$\begin{aligned} \Phi_u(s)\mathbb{1} &= K(s)(sI - A - BK(s))^{-1}\mathbb{1} \\ &= BK(s)\frac{1}{s}\mathbb{1} = \frac{1}{s}B \cdot K(s)\mathbb{1} = 0. \end{aligned} \quad (3.34)$$

Conversely, assume that Φ_u is relative, i.e. $\Phi_u(s)\mathbb{1} = 0$. Then,

$$\begin{aligned} \mathbb{1} &= (sI - A - BK(s))(sI - A - BK(s))^{-1}\mathbb{1} \\ &= (sI - A)(sI - A - BK(s))^{-1}\mathbb{1} - B\Phi_u(s)\mathbb{1} \\ &= (sI - A)(sI - A - BK(s))^{-1}\mathbb{1}. \end{aligned}$$

Thus, using the fact that A is relative,

$$\begin{aligned} 0 &= (sI - A)^{-1}\mathbb{1} - (sI - A - BK(s))^{-1}\mathbb{1} \\ &= \frac{1}{s}\mathbb{1} - (sI - A - BK(s))^{-1}\mathbb{1}. \end{aligned}$$

Rearranging, we have that

$$\begin{aligned} s\mathbb{1} &= (sI - A - BK(s))\mathbb{1} = s\mathbb{1} - 0 - BK(s)\mathbb{1} \\ \Rightarrow &\quad BK(s)\mathbb{1} = 0 \\ \Rightarrow &\quad K(s)\mathbb{1} = 0, \end{aligned}$$

so that K is relative.

3.7.6 Completion of Proof of Theorem 3.3.2

Assume $\frac{1}{s}C(I - \Phi_u)$ is stable. Then each entry of the transfer matrix $C(I - \Phi_u(s))$ has a zero at $s = 0$. Equivalently, for each $i = 1, ..., n$,

$$C_i - C(\Phi_u)_i(0) = 0_{n\times 1} \tag{3.35}$$

where C_i is the i^{th} column of C, and $(\Phi_u)_i$ is the i^{th} column of Φ_u. Define a mapping

$$(\Phi_u)_i(0) \in \mathbb{R}^{n\times 1} \mapsto (\tilde{\Phi}_u)_i(0) \in \mathbb{R}^{(2b+1)\times 1}$$

which removes all constrained zero entries of $(\Phi_u)_i$ due to the constraint that Φ_u is TF-structured w.r.t. $\mathcal{A}^{(b)}$. Similarly define a mapping

$$C \in \mathbb{R}^{n\times n} \mapsto \tilde{C}(i) \in \mathbb{R}^{n\times(2b+1)}$$

by extracting the columns of C which correspond to the constrained zero entries of Φ_u. Then (3.35) can be rewritten as

$$\tilde{C}(i)(\tilde{\Phi}_u)_i(0) = C_i. \tag{3.36}$$

One solution $\tilde{\Phi}_i^u(0)$ of (3.36) is given by the unit basis vector e_k, where k denotes the column of $\tilde{C}(i)$ that is equal to C_i. Since C is circulant and of rank $r > (2b+1)$, the matrix $\tilde{C}(i)$ will have full column rank, so that this solution $(\tilde{\Phi}_u)_i(0)$ is unique. Thus, the solution $\Phi_u(0)$ composed of all columns $(\Phi_u)_i$ will be nonzero and will contain entries of only ones and zeros. Thus, it could not be that $\Phi_u\mathbb{1} = 0$.

3.7.7 Proof of Theorem 3.5.2

A generalization of the closed-loop controller parameterization to the higher spatial dimension setting is as follows. A spatially-invariant system K is a controller $u = Kx$ for (3.22) if and only if the closed-loop spatially-invariant systems Φ_x and Φ_u are strictly proper and satisfy

$$\begin{bmatrix} sI - T_a & -T_{b_2} \end{bmatrix} \begin{bmatrix} \Phi_x(s) \\ \Phi_u(s) \end{bmatrix} = I.$$

Then using the definition $T_c := I - \frac{1}{N^d}T_{\mathbb{1}}$, this allows (3.25) to be written as

$$\begin{aligned}
& \inf_{\Phi_u} \left\| \begin{bmatrix} T_c & 0 \\ 0 & \gamma I \end{bmatrix} \begin{bmatrix} \frac{1}{s}(I+\Phi_u) \\ \Phi_u \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \\
& \text{s.t.} \ \ \Phi_u \text{ strictly proper}, \\
& \qquad \Phi_u \text{ TF-structured w.r.t. } \mathcal{A}^{(b)}, \\
& \qquad \Phi_u T_{\mathbb{1}} = 0 \\
\geq \ & \inf_{\Phi_u} \left\| \frac{1}{s}\left(I - \frac{1}{N^d}T_{\mathbb{1}}\right)(I+\Phi_u) \right\|_{\mathcal{H}_2}^2 \\
& \text{s.t.} \ \ \Phi_u \text{ strictly proper}, \\
& \qquad \Phi_u \text{ TF-structured w.r.t. } \mathcal{A}^{(b)}, \\
& \qquad \Phi_u T_{\mathbb{1}} = 0,
\end{aligned} \tag{3.37}$$

The objective of (3.37) can be written as

$$\begin{aligned}
& \left\| \frac{1}{s}\left(I - \frac{1}{N^d}T_{\mathbb{1}}\right)(I+\Phi_u) \right\|_{\mathcal{H}_2}^2 \\
= \ & \sum_{j=0}^{N-1} \left\| \frac{1}{s}\left(\delta(j) + (\phi_u)_j - \frac{1}{N^d} - \frac{1}{N^d}\sum_i (\phi_u)_i\right) \right\|_{\mathcal{H}_2}^2 \\
\geq \ & \sum_{|j|>b} \left\| \frac{1}{s}\left((\phi_u)_j - \frac{1}{N^d}\right) \right\|_{\mathcal{H}_2}^2,
\end{aligned}$$

where the first equality follows from the definition of the $\mathcal{H}_2$ norm for spatially-invariant systems and the inequality holds as $\Phi_u T_{\mathbb{1}} = 0$ implies $\sum_i (\phi_u)_i = 0$. Since Φ_u is constrained to be TF-structured w.r.t. $\mathcal{A}^{(b)}$, $(\phi_u)_j = 0$ for all $|j| > b$. Therefore (3.37) is bounded below by:

$$\inf_{\Phi_u \in \overline{\mathcal{R}}_s} \sum_{|j|>b} \left\| \frac{1}{s}\left(0 - \frac{1}{N^d}\right) \right\|_{\mathcal{H}_2}^2 = \infty.$$

Chapter 4

Control of Spatially-Distributed Systems over Sobolev Spaces

Abstract **- We consider the LQR controller design problem for spatially-invariant systems on the real line where the state space is a Sobolev space. Such problems arise when dealing with systems describing wave or beam-bending motion. We demonstrate that the optimal state feedback is a spatial convolution operator given by an exponentially decaying kernel, thus enabling implementation with a localized architecture. We generalize analogous results for the L_2 setting, and provide a rigorous explanation of numerical results previously observed in the Sobolev space setting. The main tool we utilize is a transformation from a Sobolev to an L_2 space, which is constructed from a spectral factorization of the spatial frequency weighting matrix of the Sobolev norm. We show the equivalence of the two problems in terms of the solvability conditions of the LQR problem. As a case study, we analyze the wave equation; for this example we provide analytical expressions for the dependence of the decay rate of the optimal LQR feedback convolution kernel on wave speed and the LQR cost weights.**

Parts of this chapter are based on the following publication:
[36] - E. Jensen, J. P. Epperlein and B. Bamieh, Localization of the LQR Feedback Kernel in Spatially-Invariant Problems over Sobolev Spaces, 2020 Conference on Decision and Control (CDC) 2020, IEEE, (To Appear).

4.1 Introduction

We consider the LQR controller design problem for distributed parameter systems over the real line with fully distributed actuation, restricting to the case of spatially-invariant dynamics. We assume the underlying state space is a *Sobolev space*, which applies to e.g. systems with wave-like dynamics and more general PDEs with higher-order temporal dynamics. We note that although most real-life systems are of finite spatial extent, infinite-spatial-extent spatially-invariant systems are often useful idealizations for large but finite systems, as shown in e.g. [3, 23, 24, 37].

In the spatially-invariant setting, the optimal LQR feedback gain will be a spatial convolution operator [16], and we seek to quantify the decay rate of this convolution kernel in the Sobolev space setting. An exponentially decaying convolution kernel is desirable in practice as this will allow the control policy to be approximated by a spatial truncation of this kernel, providing an inherent degree of *localization* of the resulting controller implementation [16]. Two directions of research in this setting are i) analyzing when constraints which ensure such localization can be imposed in a tractable way, as in e.g. [8] and ii) characterizing when the *unconstrained* optimal controller will have an inherent degree of spatial localization. Chapters 2 - 3 of this dissertation have primarily focused on the first problem, and in this chapter we switch perspectives to focus instead on the second problem, which has been studied in e.g. [16, 17, 38–40].

In the case of an underlying L_2 state space, the optimal LQR feedback for spatially-invariant systems is a spatial convolution operator whose kernel decays exponentially [16]. These methods were applied to analyze the LQG controller design problem for the heat equation in [38]. [17] provided results beyond the spatially-invariant setting, analyzing the so called spatially decaying operators over an L_2 state space. Numerical results presented in [40] suggest that the exponential decay rate provided by [16] holds when the underlying state space is a Sobolev space as well. However, as emphasized in [41], a rigorous general framework in this setting has yet to be developed. This chapter takes a step toward addressing this gap in the literature.

Our main result demonstrates that any LQR problem for a spatially-invariant system over the real line with a Sobolev space as the underlying state space has an equivalent formulation over an L_2 state space. If the original Sobolev space formulation is well-posed, then the L_2 transformation will be as well. The optimal feedback for the L_2 formulation is a convolution operator whose kernel *decays exponentially*, and the optimal feedback for the original Sobolev space formulation will have the same decay rate. This procedure extends the results of [16] from the L_2 setting to a more general Sobolev space setting.

As a case study, we analyze the wave equation. We demonstrate that the optimal LQR feedback for the the wave equation formulated over a standard Sobolev space decays exponentially, when cost of state is given by a standard Sobolev norm as well as when cost of state corresponds to a mechanical measure of energy.

The rest of this chapter is structured as follows. In Section 4.3 we introduce the LQR design problem of interest. In Section 4.5, analytic formulas demonstrate that the optimal LQR feedback kernel for the wave equation formulated over a standard Sobolev space decays exponentially. In Section 4.6 we show that this decay rate holds more generally, by formulating a procedure that converts the LQR design problem for a spatially-invariant distributed parameter system over a standard Sobolev space to an equivalent problem over an L_2 space. In Section 4.7 we analyze the LQR problem for the wave equation with cost functional corresponding to a mechanical measure of energy.

4.2 Notation & Mathematical Preliminaries

Given two Hilbert spaces $\mathcal{U}$ and $\mathcal{X}$, $\mathcal{L}(\mathcal{U},\mathcal{X})$ denotes the space of linear operators from $\mathcal{U}$ to $\mathcal{X}$; to simplify notation we write $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X},\mathcal{X})$. An operator $B \in \mathcal{L}(\mathcal{U},\mathcal{X})$ is *bounded* if

$$\|B\|_{\mathcal{U}\to\mathcal{X}} := \sup_{\|u\|_{\mathcal{U}}=1} \|Bu\|_{\mathcal{X}} < \infty.$$

The domain of a linear operator $B \in \mathcal{L}(\mathcal{U},\mathcal{X})$ is denoted by $\mathcal{D}(B) \subset \mathcal{U}$, and the adjoint is denoted by $B^\dagger$, i.e.

$$\langle Bf, g\rangle_{\mathcal{X}} = \left\langle f, B^\dagger g\right\rangle_{\mathcal{U}}, \text{ for all } f \in \mathcal{D}(B),\ g \in \mathcal{D}(B^\dagger)$$

B is *self-adjoint* if $B = B^\dagger$ and $\mathcal{D}(B) = \mathcal{D}(B^\dagger)$.

$L_2^n(\mathbb{R})$ denotes the set of square-integrable functions $f : \mathbb{R} \to \mathbb{C}^n$ equipped with the inner product

$$\langle \psi, \phi\rangle_{L_2^n} := \int_{x\in\mathbb{R}} \phi^*(x)\psi(x)dx,$$

where $(^*)$ denotes the complex conjugate transpose. When dimensions are clear from context, we simply write $L_2 = L_2^n(\mathbb{R})$

Definition 4.2.1 (Weighted L_2 Space)
Let $W : \mathbb{R} \to \mathbb{R}^{n\times n}$ be a matrix-valued function of the form

$$W(\lambda) = \mathrm{diag}\{w_1(\lambda), ..., w_n(\lambda)\}, \quad w_\ell(\lambda) = \sum_{j=0}^{m_\ell} c_{\ell j}\lambda^{2j}, \tag{4.1}$$

where $c_{\ell j} \in \{0, 1\}$ for all ℓ, j and for each ℓ, $c_{\ell j} \neq 0$ for at least one j. Note that $W(\lambda)$ defined by (4.1) *is positive definite for all λ except possibly $\lambda = 0$. The corresponding* weighted L_2 space*, $L_W^n(\mathbb{R})$ is defined by*

$$\begin{aligned} L_W^n(\mathbb{R}) &:= \{f : \mathbb{R} \to \mathbb{C}^n; \|f\|_{L_W}^n < \infty\}, \\ \langle f, g\rangle_{L_W^n(\mathbb{R})} &:= \int_{\lambda\in\mathbb{R}} f^*(\lambda)W_\lambda g(\lambda)d\lambda. \end{aligned} \tag{4.2}$$

Again to simplify notation, we often write $L_W = L_W^n(\mathbb{R})$.

A *multiplication operator* is an operator $M_{\hat{b}} \in \mathcal{L}(L_W^n, L_\mathcal{V}^m)$ of the form

$$(M_{\hat{b}}f)(\lambda) := \hat{b}(\lambda)f(\lambda),$$

for a measurable function $\hat{b} : \mathbb{R} \to \mathbb{C}^{m\times n}$. $\hat{b}$ is referred to as the *symbol* of the operator $M_{\hat{b}}$, and we often denote $\hat{b}(\lambda)$ by $\hat{b}_\lambda$. The adjoint of a multiplication operator $M_{\hat{b}} \in \mathcal{L}(L_\mathcal{V}^n, L_W^m)$ is itself a multiplication operator $(M_{\hat{b}})^\dagger = M_{\hat{b}^\dagger}$ with symbol $\hat{b}^\dagger$ given by

$$(\hat{b}^\dagger)_\lambda := V_\lambda^{-1}(\hat{b}_\lambda)^* W_\lambda. \tag{4.3}$$

Definition 4.2.2 (Sobolev Space) *Let $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ be a multiindex. The* standard Sobolev space $\mathcal{H}^\alpha(\mathbb{R})$ *is defined by*

$$\begin{aligned} \mathcal{H}^\alpha(\mathbb{R}) &:= \{f : \mathbb{R} \to \mathbb{C}^n;\ \|f\|_{\mathcal{H}^\alpha(\mathbb{R})} < \infty\}, \\ \langle f, g\rangle_{\mathcal{H}^\alpha(\mathbb{R})} &:= \sum_{j=1}^{n} \sum_{\ell=0}^{\alpha_j} \left\langle \frac{\partial^\ell f}{\partial x^\ell}, \frac{\partial^\ell g}{\partial x^\ell} \right\rangle_{L_2(\mathbb{R})}. \end{aligned} \tag{4.4}$$

Unlike the standard Sobolev spaces, the homogeneous Sobolev spaces *are defined by inner products which weight derivatives of functions but possibly* not *the functions themselves. In particular, the homogeneous Sobolev space $\mathcal{H}_0^\alpha(\mathbb{R})$ is defined by*

$$\begin{aligned} \mathcal{H}_0^\alpha(\mathbb{R}) &:= \{f : \mathbb{R} \to \mathbb{C}^n;\ \|f\|_{\mathcal{H}_0^\alpha(\mathbb{R})} < \infty\}, \\ \langle f, g\rangle_{\mathcal{H}^\alpha(\mathbb{R})} &:= \sum_{j=1}^{n} \sum_{\ell=1}^{\alpha_j} \left\langle \frac{\partial^\ell f}{\partial x^\ell}, \frac{\partial^\ell g}{\partial x^\ell} \right\rangle_{L_2(\mathbb{R})}. \end{aligned} \tag{4.5}$$

A Hilbert space direct sum of homogeneous Sobolev spaces (and possibly standard Sobolev spaces) is also referred to as a homogeneous Sobolev space.

Example 4.2.1 *The space*

$$\begin{aligned} \mathcal{H}^{(1,0)}(\mathbb{R}) &:= \left\{f : \mathbb{G} \to \mathbb{R}^2;\ \|f\|^2_{\mathcal{H}^{(1,0)}(\mathbb{R})} < \infty\right\} \\ \left\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\rangle_{\mathcal{H}^{(1,0)}(\mathbb{R})} &:= \langle f_1, g_1\rangle_{L_2} + \langle f_1', g_1'\rangle_{L_2} + \langle f_2, g_2\rangle_{L_2} \end{aligned} \tag{4.6}$$

is a standard Sobolev space. The space

$$\begin{aligned} H &:= \mathcal{H}_0^1(\mathbb{R}) \oplus L_2(\mathbb{R}) = \left\{f : \mathbb{G} \to \mathbb{R}^2;\ \|f\|^2_H < \infty\right\} \\ \left\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\rangle_H &:= \langle f_1', g_1'\rangle_{L_2} + \langle f_2, g_2\rangle_{L_2} \end{aligned} \tag{4.7}$$

is a homogeneous Sobolev space. Note that two functions f, g are in the same equivalence class of this homogeneous space H if $f_2 = g_2$ a.e.[1] *and $f_1 - g_1 = c$ a.e. for some constant c.*

The Fourier transform provides a structured mapping between Sobolev spaces and weighted L_2 spaces as stated in the following proposition. We note that the following result for the case of standard Sobolev spaces was provided in [42]; the homogeneous result follows similarly.

Proposition 4.2.1 *The Fourier transform is an isometric isomorphism from the standard Sobolev space $\mathcal{H}^\alpha(\mathbb{R})$ to the weighted space L_W, with $W := \mathrm{diag}\{w_1, ..., w_n\}$, $w_\ell(\lambda) = \sum_{j=0}^{\alpha_\ell} \lambda^{2j}$, and from the homogeneous Sobolev space $\mathcal{H}_0^\alpha(\mathbb{R})$ to the weighted space L_W with $W := \mathrm{diag}\{w_1, ..., w_n\}$, $w_\ell(\lambda) = \sum_{j=1}^{\alpha_\ell} \lambda^{2j}$.*

[1] Here and throughout this chapter, almost everywhere (a.e.) is with respect to the standard Lebesgue measure on $\mathbb{R}$

Example 4.2.2 *The Fourier transform is an isometric isomorphism:*

$$\mathcal{F} : \mathcal{H}^{(1,0)}(\mathbb{R}) \to L_{\mathcal{W}},\ \mathcal{W}_\lambda = \begin{bmatrix} 1+\lambda^2 & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.8}$$

and

$$\mathcal{F} : H \to L_{\mathcal{W}},\ \mathcal{W}_\lambda = \begin{bmatrix} \lambda^2 & 0 \\ 0 & 1 \end{bmatrix}. \tag{4.9}$$

Proposition 4.2.1 allows us to identify Sobolev spaces with weighted L_2 spaces. Let $\mathcal{X}$ denote a (possibly homogeneous) Sobolev space with $\mathcal{F} : \mathcal{X} \to L_W(\mathbb{R})$. We refer to $W : \mathbb{R} \to \mathbb{R}^{n\times n}$ as the *spatial frequency weighting matrix* of $\mathcal{X}$. When $\mathcal{X}$ is a standard Sobolev space, $W(\lambda)$ is positive definite for all $\lambda \in \mathbb{R}$, and when $\mathcal{X}$ is a homogeneous Sobolev space, $W(\lambda)$ is positive definite for all $\lambda \in \mathbb{R} \setminus \{0\}$ and positive semidefinite at $\lambda = 0$.

4.3 Problem Set-up

We consider distributed parameter systems of the form

$$\begin{aligned} \partial_t\psi(x,t) &= (A\psi)(x,t) + (Bu)(x,t) + w(x,t) \\ y(x,t) &= (C\psi)(x,t) \end{aligned} \tag{4.10}$$

where the state ψ, control signal u, disturbance w and output y are functions of a spatial variable $x \in \mathbb{R}$ and a temporal variable $t \in \mathbb{R}^+ = [0,\infty)$. Lower case letters denote such (possibly vector-valued) *spatio-temporal signals*

$$\psi(x,t) \quad x \in \mathbb{R},\ t \in \mathbb{R}^+ := [0,\infty).$$

$\psi(x,t)$ is the value of the signal ψ at time t and spatial location x. For a fixed time t, the functions $\psi(\cdot,t)$ and $u(\cdot,t)$, denoted as $\psi(t)$ and $u(t)$, represent a spatially distributed signal. We use a ($\hat{}$) to denote the *spatial Fourier transform* of a spatio-temporal signal:

$$\begin{aligned} \hat{\psi}(\lambda,t) &:= (\mathcal{F}\psi)(\lambda,t) \\ &:= \frac{1}{2\pi}\int_{x\in\mathbb{R}} \psi(x,t)e^{-i\lambda x}dx, \quad \lambda \in \mathbb{R},\ t \in \mathbb{R}^+, \end{aligned} \tag{4.11}$$

and we denote the signals $\hat{\psi}(t) = \hat{\psi}(\cdot,t)$, and $\hat{\psi}_\lambda = \hat{\psi}(\lambda,\cdot)$. Note that this transform (4.11) is taken only in the spatial variable x; the temporal variable t remains in the original space.

We consider the design of a state-feedback control policy $u(x,t) = (K\psi)(x,t)$ for systems (4.10), noting that actuation over the continuous domain $\mathbb{R}$ is an idealized assumption and actuation will be implemented in practice with some degree of discretization. This is often a useful approximation, just as continuous time approximations are often used for fast temporally sampled systems.

To ensure solutions are well-defined, we make the common assumption that A generates a C_0-semigroup $\{e^{At}\}$ [43] on $\mathcal{X}$ with domain $\mathcal{D}(A)$ dense in $\mathcal{X}$. We also assume that $u(t) \in \mathcal{D}(B) \subset \mathcal{U}$, $e^{At}B \in \mathcal{L}(\mathcal{U},\mathcal{H})$ is bounded for each $t \geq 0$, and $C \in \mathcal{L}(\mathcal{X},\mathcal{Y})$ is bounded.

Example 4.3.1 *The dynamics of the undamped wave equation over the real line with fully distributed actuation u are described by the PDE*

$$\partial_t^2 \xi(x,t) \;=\; c^2 \partial_x^2 \xi(x,t) + \frac{1}{\rho} u(x,t), \tag{4.12}$$

where $c > 0$ is the wave speed. Defining $\psi(t) := \begin{bmatrix} \xi(\cdot,t) & \partial_t \xi(\cdot,t) \end{bmatrix}^T$, we write (4.12) *in* state space form (4.10) *as*

$$\frac{d}{dt}\psi(t) = \begin{bmatrix} 0 & I \\ c^2 \partial_x^2 & 0 \end{bmatrix} \psi(t) + \begin{bmatrix} 0 \\ \frac{1}{\rho} I \end{bmatrix} u(t) \quad =: A\psi(t) + Bu(t) \tag{4.13}$$

A generates a C_0-semigroup $\{e^{At}\}$ on the Sobolev space $\mathcal{X} := \mathcal{H}^{(1,0)}(\mathbb{R})$, and $B \in \mathcal{L}(L_2, \mathcal{X})$ is a bounded operator. Recall from Definition 4.2.2 that

$$\langle f, g \rangle_{\mathcal{H}^{(1,0)}(\mathbb{R})} = \left\langle \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}, \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} \right\rangle_{\mathcal{H}^{(1,0)}(\mathbb{R})} := \langle f_1, g_1 \rangle_{L_2} + \langle f_1', g_1' \rangle_{L_2} + \langle f_2, g_2 \rangle_{L_2}. \tag{4.14}$$

We remark that this choice of A and B was employed in e.g. [40, 41], though many other state space representations exist. (One such alternate state space representation, over an L_2 space, will be provided in Section 4.6).

We design a state feedback control policy $u = K\psi$ for systems (4.10). Of course this control policy should be designed so that the resulting closed-loop system is stable, and we additionally choose the control to optimize a quadratic measure of performance. Before formally stating the optimal control design problem of interest, we begin by presenting a review of stability in this infinite dimensional setting.

Stability

Definition 4.3.1 *Let A generate a C_0-semigroup of bounded operators $\{e^{At}\}$ on a Hilbert space $\mathcal{X}$, and assume $\mathcal{D}(A)$ is dense in $\mathcal{X}$. Then the system*

$$\partial_t \psi(t) = A\psi(t) \tag{4.15}$$

is exponentially stable *if there exist constants $M, \alpha > 0$ for which*

$$\|e^{At}\| \leq M e^{-\alpha t}, \quad \textit{for all } t \geq 0, \tag{4.16}$$

where $\|\cdot\|$ denotes the $\mathcal{X} \to \mathcal{X}$ induced operator norm, i.e.

$$\|e^{At}\| = \sup_{f \in \mathcal{X}\setminus\{0\}} \frac{\|e^{At} f\|_{\mathcal{X}}}{\|f\|_{\mathcal{X}}}. \tag{4.17}$$

The state feedback policy $u = K\psi$ for system (4.10) *is said to be* exponentially stabilizing *if the closed-loop system*

$$\partial_t \psi(x,t) = (A + BK)\psi(x,t) =: A_{\mathrm{cl}} \psi(x,t)$$

is exponentially stable.

The following proposition provides a Lyapunov condition for exponential stability; this result was taken from [44, Lem. 4.3.3].

Proposition 4.3.1 *The system* (4.15) *is exponentially stable if and only if there exists a bounded positive-definite operator* $\Pi = \Pi^\dagger \in \mathcal{L}(\mathcal{X})$ *which solves the Lyapunov equation*

$$\langle A\psi, \Pi\psi\rangle_{\mathcal{X}} + \langle \Pi\psi, A\psi\rangle_{\mathcal{X}} + \langle \psi, \psi\rangle_{\mathcal{X}} = 0, \quad \textit{for all } \psi \in \mathcal{D}(A). \tag{4.18}$$

Throughout this chapter we use the following shorthand notation to write the Lyapunov condition (4.18) together with its domain:

$$\Pi A + A^\dagger \Pi + I = 0. \tag{4.19}$$

We are now able to formally state the optimal controller design problem analyzed in this chapter.

LQR Controller Design

We consider the optimal LQR controller design problem for distributed parameter systems (4.10):

$$\begin{aligned} \inf_{u=K\psi} \quad & \int_0^\infty \langle y(t), y(t)\rangle_{\mathcal{Y}} + \langle u(t), Ru(t)\rangle_{\mathcal{U}} \; dt \\ \text{s.t.} \quad & \text{dynamics (4.10)} \end{aligned} \tag{4.20}$$

where $R : \mathcal{D}(R) \subset \mathcal{U} \to \mathcal{U}$ is a positive-definite operator specifying the cost of control, and $y(t) = C\psi(t)$ defines a performance output to optimize. Note that

$$\langle y(t), y(t)\rangle_{\mathcal{Y}} = \langle C\psi(t), C\psi(t)\rangle_{\mathcal{Y}} = \left\langle \psi(t), C^\dagger C\psi(t) \right\rangle_{\mathcal{X}}, \tag{4.21}$$

so that $Q := C^\dagger C \in \mathcal{L}(\mathcal{X})$ specifies the cost of state in the standard LQR problem. We restrict $Q : \mathcal{D}(Q) \subset \mathcal{X}$ to be a positive semi-definite bounded operator.

Definition 4.3.2 *We say the LQR problem* (4.20) *is* well-posed *if the following conditions hold:*

1. *For each* t*,* $e^{At}B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ *is a bounded operator,*
2. (A, B) *is* exponentially stabilizable*,*
3. (C, A) *is* exponentially detectable.

When (4.20) is well-posed, there exists a unique exponentially stabilizing solution $u = K\psi$ to (4.20). The notions of exponential stabilizability and exponential detectability provided in conditions (2) and (3) are formally defined as follows.

Definition 4.3.3 *Let* A *generate a* C_0*-semigroup of bounded operators on* $\mathcal{X}$ *and* $B : \mathcal{U} \to \mathcal{X}$*.* (A, B) *is exponentially stabilizable if there exists a bounded operator* $F : \mathcal{X} \to \mathcal{U}$ *such that* $(A - BF)$ *is exponentially stable.* (C, A) *is exponentially detectable if* $(A^\dagger, C^\dagger)$ *is exponentially stabilizable.*

The following proposition provides a check for stabilizability via an operator Riccati equation; this result comes from [45].

Proposition 4.3.2 *Let A generate a C_0-semigroup of bounded operators on $\mathcal{X}$ and $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$. Then the pair (A, B) is stabilizable if and only if there exists a bounded positive definite solution $\Pi = \Pi^\dagger \in \mathcal{L}(\mathcal{X})$ to the operator Riccati equation*

$$\langle \Pi f, Ag \rangle_{\mathcal{X}} + \langle Af, \Pi g \rangle_{\mathcal{X}} + \langle f, g \rangle_{\mathcal{X}} - \left\langle \Pi B B^\dagger \Pi f, g \right\rangle_{\mathcal{X}} = 0, \quad \text{for all } f, g \in \mathcal{D}(A). \tag{4.22}$$

Throughout this chapter, we use the following shorthand notation to write an operator Riccati equation of the form (4.22) together with its domain:

$$\Pi A + A^\dagger \Pi + I - \Pi B B^\dagger \Pi = 0. \tag{4.23}$$

Example 4.3.2 *We design a feedback control policy u for the wave equation with state space model* (4.13) *to optimize an LQR objective*

$$\int_0^\infty \langle \psi(t), Q\psi(t) \rangle_{\mathcal{X}} + \langle u(t), Ru(t) \rangle_{\mathcal{U}} \;\, dt, \tag{4.24}$$

where $R = \gamma I$ is a multiple of the identity operator on $\mathcal{U} = L^2(\mathbb{R})$ and Q is the identity operator on $\mathcal{X} = \mathcal{H}^{(1,0)}(\mathbb{R})$. It was observed numerically in [42] that this choice of LQR cost functional led to an exponentially decaying optimal feedback, and in this chapter we formalize those observations.

Throughout this chapter, we consider systems (4.10) for which $\mathcal{X}$, $\mathcal{U}, \mathcal{Y}$ are all Sobolev spaces. We additionally restrict our attention to systems that are *spatially-invariant.*

4.4 Spatially-Invariant Systems

Definition 4.4.1 *An operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ is a* spatial convolution operator *if it can be written in the form*

$$(B\psi)(x) = (b * \psi)(x) = \int_{y \in \mathbb{G}} b(x-y)\psi(y)dy \tag{4.25}$$

for some convolution kernel *b. We allow for e.g. dirac delta functions in the convolution kernels so that e.g. point evaluation and spatial differential operators can be represented in this form. When A, B and C of* (4.10) *are all spatial convolution operators, we say that the system* (4.10) *is a* spatially-invariant system.

A spatial convolution operator B is 'diagonalized' by a spatial Fourier transform to a multiplication operator [16], i.e.

$$\mathcal{F} B \mathcal{F}^{-1} =: M_{\hat{b}},$$

where $\hat{b} = \mathcal{F} b$ the Fourier transform of the convolution kernel. $\|B\| = \|M_{\hat{b}}\|$, and B is self-adjoint if and only if $M_{\hat{b}}$ is.

Example 4.4.1 *The state equation of the wave equation example* (4.13) *is spatially-invariant and is diagonalized as:*

$$\partial_t \hat{\psi}(\lambda, t) = \begin{bmatrix} 0 & 1 \\ -c^2\lambda^2 & 0 \end{bmatrix} \hat{\psi}(\lambda, t) + \begin{bmatrix} 0 \\ \frac{1}{\rho} \end{bmatrix} \hat{u}(\lambda, t) =: \hat{a}_\lambda \hat{\psi}_\lambda(t) + \hat{b}_\lambda \hat{u}_\lambda(t). \tag{4.26}$$

Definition 4.4.2 *A spatial convolution operator B is said to* decay exponentially with rate $\tilde{\beta} > 0$ *if its defining convolution kernel b satisfies*

$$b(x)e^{\tilde{\beta}|x|} \to 0 \text{ as } |x| \to \infty.$$

Definition 4.4.3 *Given a multiplication operator $M_{\hat{b}}$, a function $\hat{b}_e$ on the complex plane which recovers the function $\hat{b}$ when restricted to the imaginary axis, i.e.*

$$\hat{b}_e(\sigma)\Big|_{\sigma=i\lambda} = \hat{b}(\lambda), \tag{4.27}$$

is said to be an extension *of the symbol $\hat{b}$ to the complex plane.*

Employing [46, Thm 7.4.2], if there exists an extension $\hat{b}_e$ of $\hat{b}$ which is analytic and satisfies a polynomial growth bound on the strip

$$\Gamma_\beta := (-\beta, \beta) + i\mathbb{R} \ \subset \mathbb{C}, \tag{4.28}$$

then the inverse Fourier transform, b, of $\hat{b}$ is decays exponentially with rate $\tilde{\beta}$ i.e.

$$|b(x)|e^{\tilde{\beta}|x|} \underset{|x|\to\infty}{\longrightarrow} 0, \tag{4.29}$$

for any $\tilde{\beta} < \beta$.

Note that there may be more than one choice of extension for a given symbol. It is not necessary that every possible extension is analytic in some strip to ensure an exponential decay rate, just that there exists one extension which satisfies this requirement.

The following proposition is a slight modification of [42, Lem. 3.5].

Proposition 4.4.1 *Let $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ be a spatially-invariant operator between the two (possibly homogeneous) Sobolev spaces $\mathcal{U}$ and $\mathcal{X}$ with spatial frequency weighting matrices V and W, respectively. Let $\hat{b}$ denote the symbol of the corresponding multiplication operator $M_{\hat{b}} : L_V \to L_W$. The adjoint $M_{\hat{b}}^\dagger$ is also a multiplication operator with symbol $\hat{b}^\dagger$ given by*

$$\hat{b}_\lambda^\dagger := V_\lambda^{-1} \hat{b}_\lambda^* W_\lambda \tag{4.30}$$

for a.e. $\lambda \in \mathbb{R}$.

In the spatially-invariant setting, we provide a check for stabilizability and detectability in the spatial frequency domain.

4.4.1 Stability

Consider the Lyapunov condition for stability provided in Proposition 4.3.1. Assume A is a spatial convolution operator which generates a C_0-semigroup of bounded operators on a (standard or homogeneous) Sobolev space $\mathcal{X}$ and let W denote the spatial frequency weighting matrix of $\mathcal{X}$. In this case, the Lyapunov equation (4.18) can be rewritten in the spatial frequency domain as

$$\int_{\lambda\in\mathbb{R}} \hat{\psi}_\lambda^* \Big(\hat{a}_\lambda W_\lambda \hat{\pi}_\lambda + \hat{\pi}_\lambda^* W_\lambda \hat{a}_\lambda + W_\lambda\Big)\hat{\psi}_\lambda \, d\lambda = 0 \quad \text{for all } \hat{\psi} \in \mathcal{D}(M_{\hat{a}}). \tag{4.31}$$

Note that W_λ is positive definite for all $\lambda \neq 0$ since $\mathcal{X}$ is a Sobolev space. In the case that $\mathcal{X}$ is a *standard* Sobolev space, W_0 is positive definite as well, while W_0 is only positive semi-definite in the homogeneous Sobolev space setting. Any $\psi \in \mathcal{X}$ can be written as

$$\psi(x) = \overline{\psi} \; + \; \tilde{\psi}(x), \tag{4.32}$$

where $\overline{\psi}$ is the spatial DC component. Equivalently, in the spatial frequency domain

$$\hat{\psi}(\lambda) = \overline{\psi}\;\delta(\lambda) \; + \; \hat{\tilde{\psi}}(\lambda). \tag{4.33}$$

Thus, (4.31) may be written as

$$\int_{\lambda\in\mathbb{R}\setminus\{0\}} \hat{\tilde{\psi}}_\lambda^* \Big(\hat{a}_\lambda W_\lambda \hat{\pi}_\lambda + \hat{\pi}_\lambda^* W_\lambda \hat{a}_\lambda + W_\lambda\Big)\hat{\tilde{\psi}}_\lambda \, d\lambda \; + \; \overline{\psi}^* \Big(\hat{a}_0 W_0 \hat{\pi}_0 + \hat{\pi}_0^* W_0 \hat{a}_0 + W_0\Big)\overline{\psi} = 0 \tag{4.34}$$

for all $\hat{\psi} \in \mathcal{D}(M_{\hat{a}})$.

In the case that $\mathcal{X}$ is a *standard* Sobolev space, i.e. $\mathcal{X} = \mathcal{H}^\alpha(\mathbb{R})$ for some multiindex α, the spatial DC component of (4.32) will be $\overline{\psi} = 0$ since $\psi \in \mathcal{X} = \mathcal{H}^\alpha(\mathbb{R})$ implies that $\int_{x\in\mathbb{R}} \psi^*(x)\psi(x)dx < \infty$. Then (4.31) may be reduced to

$$\int_{\lambda\in\mathbb{R}\setminus\{0\}} \hat{\psi}_\lambda^* \Big(\hat{a}_\lambda W_\lambda \hat{\pi}_\lambda + \hat{\pi}_\lambda^* W_\lambda \hat{a}_\lambda + W_\lambda\Big)\hat{\psi}_\lambda \, d\lambda \; = 0 \text{ for all } \hat{\psi} \in \mathcal{D}(M_{\hat{a}}). \tag{4.35}$$

In the case of a homogeneous space however, this DC component may be nonzero so the spatial frequency $\lambda = 0$ case must be explicitly checked.

The boundedness of the operator Π is equivalent to the boundedness of $M_{\hat{\pi}} = \mathcal{F}\Pi\mathcal{F}^{-1}$, which is characterized as follows. A similar result for the case of a *standard* Sobolev space is provided in [42, Cor. 3.7].

Proposition 4.4.2 *Let $\mathcal{X}$ denote a (possibly homogeneous) Sobolev space with spatial frequency weighting matrix W and let $G \in \mathcal{L}(\mathcal{X})$. be a spatial convolution operator with convolution kernel g. The operator norm of G may be computed by*

$$\|G\| = \|M_{\hat{g}}\| = \operatorname*{ess\,sup}_{\lambda\in\mathbb{R}\setminus\{0\}} \; \sigma_{\max}\left(\hat{s}_\lambda \hat{g}_\lambda \hat{s}_\lambda^{-1}\right), \tag{4.36}$$

where $W_\lambda = \hat{s}_\lambda^* \hat{s}_\lambda$ is a spectral factorization.

Proof: We employ a proof technique similar to that of [42]. Let $\psi \in \mathcal{D}(G) \subset \mathcal{X}$ be decomposed as in (4.32). Compute

$$\begin{aligned}
\|G\psi\|_{\mathcal{X}}^2 &= \|M_{\hat{g}}\hat{\psi}\|_{L_W}^2 \\
&= \left\langle M_{\hat{g}}\hat{\psi}, M_{\hat{g}}\hat{\psi}\right\rangle_{L_W} \\
&= \int_{\lambda\in\mathbb{R}} \hat{\psi}_\lambda^* \hat{g}_\lambda^* W_\lambda \hat{g}_\lambda \hat{\psi}_\lambda d\lambda \\
&= \overline{\psi}^* \hat{g}_0^* W_0 \hat{g}_0 \overline{\psi} \;+\; \int_{\lambda\in\mathbb{R}\setminus\{0\}} \hat{\tilde{\psi}}_\lambda^* \hat{s}_\lambda^* \left(\hat{s}_\lambda \hat{g}_\lambda \hat{s}_\lambda^{-1}\right)^* \left(\hat{s}_\lambda \hat{g}_\lambda \hat{s}_\lambda^{-1}\right) \hat{s}_\lambda \hat{\tilde{\psi}}_\lambda d\lambda \\
&\overset{(1)}{=} \int_{\lambda\in\mathbb{R}\setminus\{0\}} \hat{\tilde{\psi}}_\lambda^* \hat{s}_\lambda^* \left(\hat{s}_\lambda \hat{g}_\lambda \hat{s}_\lambda^{-1}\right)^* \left(\hat{s}_\lambda \hat{g}_\lambda \hat{s}_\lambda^{-1}\right) \hat{s}_\lambda \hat{\tilde{\psi}}_\lambda d\lambda \\
&\leq \left(\operatorname*{ess\,sup}_{\lambda\in\mathbb{R}\setminus\{0\}} \sigma_{\max}\{\hat{s}_\lambda \hat{g}_\lambda \hat{s}_\lambda^{-1}\} \cdot \|\hat{s}\hat{\tilde{\psi}}\|_{L_2}\right)^2 \\
&= \left(\operatorname*{ess\,sup}_{\lambda\in\mathbb{R}\setminus\{0\}} \sigma_{\max}\{\hat{s}_\lambda \hat{g}_\lambda \hat{s}_\lambda^{-1}\} \cdot \|\hat{\tilde{\psi}}\|_{L_W}\right)^2 \\
&\leq \left(\operatorname*{ess\,sup}_{\lambda\in\mathbb{R}\setminus\{0\}} \sigma_{\max}\{\hat{s}_\lambda \hat{g}_\lambda \hat{s}_\lambda^{-1}\} \cdot \|\hat{\psi}\|_{L_W}\right)^2,
\end{aligned} \tag{4.37}$$

where equality (1) follows from the fact that the $\overline{\psi}$ is zero when $\mathcal{X}$ is a standard Sobolev space, and when $\mathcal{X}$ is a homogeneous Sobolev space, and nonzero entry of $\overline{\psi}$ will correspond to a zero entry of W_0. ■

The preceding analysis and Proposition 4.4.2 provide an explicit condition for stability in the case of a state space $\mathcal{X}$ that is a standard Soblev space $\mathcal{H}^\alpha(\mathbb{R})$ or a homogeneous Sobolev space $\mathcal{H}_0^\alpha(\mathbb{R})$, which is stated in the following theorem. We remark that this result for the case of a *standard* Sobolev space is provided in [42, Thm. 3.10].

Theorem 4.4.3 *The operator $A \in \mathcal{L}(\mathcal{X})$ is exponentially stable if and only if the following hold:*

1. *There exists a solution $\hat{\pi}_\lambda = \hat{\pi}_\lambda^* \succ 0$ of the Lyapunov equation*

$$\hat{a}_\lambda W_\lambda \hat{\pi}_\lambda + \hat{\pi}_\lambda^* W_\lambda \hat{a}_\lambda + W_\lambda = 0, \quad \textit{for} \begin{cases} \textit{a.e. } \lambda \in \mathbb{R}, & \mathcal{X} = \mathcal{H}^\alpha(\mathbb{R}) \\ \lambda = 0 \;\&\; \textit{a.e. } \lambda \in \mathbb{R}\setminus\{0\}, & \mathcal{X} = \mathcal{H}_0^\alpha(\mathbb{R}), \end{cases} \tag{4.38}$$

   where W is the spatial frequency weighting matrix of $\mathcal{X}$.

2. *This solution $\hat{\pi}$ is the symbol of a* bounded *multiplication operator, i.e.*

$$\|M_{\hat{\pi}}\| < \infty \tag{4.39}$$

   where $\|M_\pi\|$ is computed as in Proposition 4.4.2.

It is straightforward to generalize this result to homogeneous Sobolev spaces that aren't in the specific form of $\mathcal{H}_0^\alpha$, these details are omitted.

4.4.2 Stabilizability & Detectability

Following the same ideas as Theorem 4.4.3, we derive the following explicit frequency domain conditions for stabilizability and detectability in the case of a spatially-invariant system over a state space $\mathcal{X}$ that is a Sobolev space.

Theorem 4.4.4 *Let A generate a C_0-semigroup on $\mathcal{X}$ and $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$, where $\mathcal{X} = \mathcal{H}^\alpha(\mathbb{R})$ is a standard Sobolev space and $\mathcal{U}$ is a (possibly homogeneous) Sobolev space. Let W and V denote the spatial frequency weighting matrices of $\mathcal{X}$ and $\mathcal{U}$ respectively. Then (A, B) is exponentially stabilizable if and only if*

1. *There exists a solution $\hat{p}_\lambda = \hat{p}_\lambda^* \succ 0$ of the matrix Riccati equation*

$$\hat{p}_\lambda \hat{a}_\lambda + \hat{a}_\lambda^* \hat{p}_\lambda + W_\lambda - \hat{p}_\lambda \hat{b}_\lambda V_\lambda^{-1} \hat{b}_\lambda^* \hat{p}_\lambda = 0 \tag{4.40}$$

   for a.e. $\lambda \in \mathbb{R} \setminus \{0\}$

2. *and this solution satisfies the boundedness condition*

$$\operatorname*{ess\,sup}_{\lambda \in \mathbb{R}\setminus\{0\}} \sigma_{\max}\{\hat{s}_\lambda^{-*} p_\lambda \hat{s}_\lambda^{-1}\}, \tag{4.41}$$

   where $W_\lambda = \hat{s}_\lambda^* \hat{s}_\lambda$ is a spectral factorization.

Proof: Any $\psi, \phi \in \mathcal{D}(A) \subset \mathcal{X}$ can be decomposed as in (4.32) as

$$\begin{aligned} \psi(t) &= \tilde{\psi}(t) + \overline{\psi}\delta(t) \\ \phi(t) &= \tilde{\phi}(t) + \overline{\phi}\delta(t) \end{aligned} \tag{4.42}$$

where $\overline{\psi} = \overline{\phi} = 0$ and when $\mathcal{X}$ is a standard Sobolev space. From Proposition 4.3.2, stabilizability is equivalent to the existence of a bounded solution to the Riccati equation (4.22). In this setting, this Riccati equation can be written in the spatial frequency domain as

$$0 = \int_{\lambda \in \mathbb{R}\setminus\{0\}} \hat{\tilde{\psi}}_\lambda^* \left(\hat{a}_\lambda^* W_\lambda \hat{\pi}_\lambda + \hat{\pi}_\lambda W_\lambda \hat{a}_\lambda + W_\lambda - W_\lambda \hat{b}_\lambda V_\lambda^{-1} \hat{b}_\lambda^* W_\lambda \hat{\pi}_\lambda \right) \hat{\tilde{\phi}}_\lambda d\lambda \text{ for all } \psi, \phi \in \mathcal{D}(A) \subset \mathcal{X} \tag{4.43}$$

Define $\hat{\pi}_\lambda := W_\lambda^{-1} \hat{p}_\lambda$, and let Π denote the spatially-invariant operator with convolution kernel $\pi = \mathcal{F}^{-1}\hat{\pi}$. Then $\Pi = \Pi^\dagger$ and from (4.40), we see that Π satisfies the operator Riccati equation (4.23). By Proposition 4.4.2, boundedness of Π can be checked with the condition

$$\begin{aligned} \|\Pi\| &= \operatorname*{ess\,sup}_{\lambda \in \mathbb{R}\setminus\{0\}} \sigma_{\max}\{\hat{s}_\lambda \hat{\pi}_\lambda \hat{s}_\lambda^{-1}\} \\ &= \operatorname*{ess\,sup}_{\lambda \in \mathbb{R}\setminus\{0\}} \sigma_{\max}\{\hat{s}_\lambda W_\lambda^{-1} \hat{p}_\lambda \hat{s}_\lambda^{-1}\} \\ &= \operatorname*{ess\,sup}_{\lambda \in \mathbb{R}\setminus\{0\}} \sigma_{\max}\{\hat{s}_\lambda^{-*} \hat{p}_\lambda \hat{s}_\lambda^{-1}\} \qquad < \infty. \end{aligned} \tag{4.44}$$

■

In the case of a homogeneous Sobolev space $\mathcal{X}$, the components $\overline{\psi}$ and $\overline{\phi}$ may be nonzero, and the spatial frequency $\lambda = 0$ component must be explicitly checked. In particular, Theorem 4.4.4 can be modified to hold in the case that $\mathcal{X}$ is a homogeneous Sobolev space and $\mathcal{U}$ is a standard Sobolev space by replacing condition (1) with the condition that (4.40) holds for a.e. $\lambda \in \mathbb{R} \backslash \{0\}$ *and* for $\lambda = 0$.

An analogous result for detectability is stated in the following theorem, whose proof is omitted.

Theorem 4.4.5 *Let A generate a C_0-semigroup on $\mathcal{X}$ and $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ is a standard Sobolev space and $\mathcal{Y}$ is a (possibly homogeneous) Sobolev space with spatial frequency weighting matrices W and V. Then (C, A) is exponentially detectable if and only if*

1. *There exists a solution $\hat{p}_\lambda = \hat{p}_\lambda^* \succ 0$ of the matrix Riccati equation*
$$\hat{p}_\lambda \hat{a}_\lambda^* + \hat{a}_\lambda \hat{p}_\lambda + W_\lambda^{-1} - \hat{p}_\lambda \hat{c}_\lambda^* V_\lambda \hat{c}_\lambda \hat{p}_\lambda = 0 \tag{4.45}$$
for a.e. $\lambda \in \mathbb{R} \setminus \{0\}$

2. *and this solution satisfies the boundedness condition*
$$\operatorname*{ess\,sup}_{\lambda \in \mathbb{R} \backslash 0} \sigma_{\max}\{\hat{s}_\lambda p_\lambda \hat{s}_\lambda^*\} < \infty \tag{4.46}$$
where $W_\lambda = \hat{s}_\lambda^* \hat{s}_\lambda$ is a spectral factorization.

Theorems 4.4.4 and 4.4.5 allow for a straightforward procedure of checking conditions (2) and (3) for well-posedness of a spatially-invariant LQR problem in the spatial frequency domain. In particular, to confirm these conditions one only needs to check the existence and boundedness of the solution of two parameterized families of *finite-dimensional* Riccati equations.

4.4.3 LQR Controller Design for Spatially-Invariant Systems

Consider the LQR problem (4.20) for system (4.10) for which A generates a C_0-semigroup of bounded operators on a standard or homogeneous Sobolev space $\mathcal{X}$, $u(t) \in \mathcal{U}$ and $y(t) \in \mathcal{Y}$ for standard Sobolev spaces $\mathcal{U}$ and $\mathcal{Y}$. Let W and V denote the spatial frequency weighting matrices of $\mathcal{X}$ and $\mathcal{U}$ respectively, and define $Q := C^\dagger C \in \mathcal{L}(\mathcal{X})$ where $y(t) := C\psi(t)$ is the performance output of interest.

Theorem 4.4.6 *The optimal solution to the LQR design problem:*
$$\begin{aligned} \inf_{u = K\psi} \quad & \int_0^\infty \langle y(t), y(t)\rangle_{\mathcal{Y}} + \langle u(t), Ru(t)\rangle_{\mathcal{U}} \ dt \\ s.t. \quad & \text{dynamics (4.10)} \end{aligned} \tag{4.47}$$
is given by the feedback
$$u = K\psi,$$
where $K \in \mathcal{L}(\mathcal{X},\mathcal{U})$ is a spatial convolution operator that is defined in the spatial frequency domain by
$$\hat{u} = M_{\hat{k}} \hat{\psi}, \ \hat{k}_\lambda = -\hat{r}_\lambda^{-1} V_\lambda^{-1} \hat{b}_\lambda^* \hat{p}_\lambda \tag{4.48}$$

where $\hat{p}_\lambda = \hat{p}_\lambda^* \succ 0$ *is the solution to the matrix Riccati equation*

$$\hat{p}_\lambda \hat{a}_\lambda + \hat{a}_\lambda^* \hat{p}_\lambda + W_\lambda \hat{q}_\lambda - \hat{p}_\lambda \hat{b}_\lambda \hat{r}_\lambda^{-1} V_\lambda^{-1} \hat{b}_\lambda^* \hat{p}_\lambda = 0 \tag{4.49}$$

for a.e. $\lambda \in \mathbb{R} \setminus \{0\}$*. When the LQR problem is well-posed (according to Definition 4.3.2), this control policy* K *is exponentially stabilizing.*

Theorem 4.4.6 is a slight modification of [42, Thm. 3.13] and its proof is similar to that of Theorem 4.4.4.

The decay rate of (4.48) provides the degree of *localization* of the optimal distributed control policy. It has been shown that (4.48) *decays exponentially*, under appropriate assumptions on A, B, Q, R, in the special case that $\mathcal{X} = L_2$ [16]. Numerical results suggest such decay rates also hold for more general choice of Sobolev space $\mathcal{X}$ [40]. The following analysis works toward rigorously proving these observed decay rates.

4.5 Application: Wave Equation

We first analyze the decay rate of the LQR feedback for the wave equation (4.12). A spatial Fourier transform converts (4.12) to a parameterized family over $\lambda \in \mathbb{R}$ of finite-dimensional dynamics:

$$\partial_t \hat{\psi}(\lambda, t) = \begin{bmatrix} 0 & 1 \\ -c^2\lambda^2 & 0 \end{bmatrix} \hat{\psi}(\lambda, t) + \begin{bmatrix} 0 \\ \frac{1}{\rho} \end{bmatrix} \hat{u}(\lambda, t) =: a_\lambda \hat{\psi}(\lambda, t) + b_\lambda \hat{u}(\lambda, t). \tag{4.50}$$

It is known [40] that A and B of (4.50) satisfy conditions (1) and (2) for well-posedness (Definition 4.3.2), i.e. (A, B) is exponentially detectable, and for each t, $e^{At}B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ is a bounded operator. For a spatially-invariant operator $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$, with (A, C) exponentially detectable, the corresponding LQR problem (4.20) is well-posed and the optimal control policy K is stabilizing.

4.5.1 Analytic Solution of Optimal Feedback Gain

It was shown in [40] that the choice of cost of state $Q = C^\dagger C = I$ is such that (Q, A) is exponentially detectable. The optimal feedback for cost of control weight $R = I$ was numerically plotted in [40] and observed to decay exponentially. Here, we analytically compute the optimal feedback for control cost weight $R = \gamma I$ and formalize this observation:

$$K_\lambda = \frac{-1}{\gamma} \begin{bmatrix} -\gamma c^2 \lambda^2 + f(\lambda) & \sqrt{\gamma(1 - 2\gamma c^2\lambda^2 + 2f(\lambda))} \end{bmatrix} \tag{4.51}$$

where $f(\lambda) := \sqrt{c^4\lambda^4\gamma^2 + \gamma(1+\lambda^2)}$, and $h(\lambda) := \frac{1}{\gamma} f(\lambda)\sqrt{\gamma(1 - 2c^2\gamma\lambda^2 + 2f(\lambda))}$. We compute the extension

$$K_e(\sigma) = \frac{-1}{\gamma} \begin{bmatrix} \gamma c^2 \sigma^2 + f_e(\sigma) & \sqrt{\gamma(1 + 2\gamma c^2\sigma^2 + 2f_e(\sigma))} \end{bmatrix}$$

where $f_e(\sigma) := \sqrt{c^4\sigma^4\gamma^2 + \gamma(1-\sigma^2)}$. The branch points [47] of the multivalued function f_e are given by $\sigma = \infty$ along with the zeros of the function $g(\sigma) := c^4\sigma^4\gamma^2 + \gamma(1-\sigma^2)$ which are given by

$$\begin{cases} \sigma = \pm\sqrt{\frac{1}{2\nu}\left(1 \pm \sqrt{1-4\nu}\right)}, & 0 < \nu \leq \frac{1}{4} \\ \sigma = \pm\left(\frac{1}{2\sqrt{\nu}}\right)\left(\sqrt{2\sqrt{\nu}+\alpha^2} \pm i\sqrt{2\sqrt{\nu}-1}\right), & \nu > \frac{1}{4}, \end{cases}$$

where

$$\nu := c^4\gamma. \tag{4.52}$$

For $\nu < \frac{1}{4}$, there are 4 distinct real-valued zeros of $g(\cdot)$; for $\nu = \frac{1}{4}$ there are 2 repeated real-valued zeros; for $\nu > \frac{1}{4}$ there are 4 complex-valued zeros (2 distinct complex conjugate pairs). The locations of the branch points in each of these 3 regimes is illustrated in Figure 4.1. The shaded region is the strip Γ_β, with β the magnitude of the real part of the zeros. When there are 4 distinct real-valued zeros, β is the smaller of the 2 real component magnitudes. A precise formula for Γ_β as a function of the parameter $\nu = c^4\gamma$ is

$$\Gamma_\beta := \begin{cases} \left\{|\mathrm{Re}(z)| < \sqrt{\frac{1}{2\nu}(1-\sqrt{1-4\nu})}\right\}, & \nu \in (0, \frac{1}{4}] \\ \left\{|\mathrm{Re}(z)| < \frac{\sqrt{2\sqrt{\nu}+1}}{2\sqrt{\nu}}\right\}, & \nu > \frac{1}{4} \end{cases} \tag{4.53}$$

K_e has no additional branch points in Γ_β, and can be uniquely defined as an analytic function in this region. β is dependent on the LQR cost parameters and the wave speed: $\beta \to 1$ for $\nu \ll \frac{1}{4}$, $\beta \to \sqrt{2}$ as $\nu \to \frac{1}{4}$, and $\beta \to 0$ for $\nu \gg \frac{1}{4}$. The largest region of analyticity (fastest rate of decay) will occur at $\nu = \frac{1}{4}$ and is given by $\beta = \sqrt{2}$. The mapping $\nu \mapsto \beta$ is non-differentiable at the point $\nu = \frac{1}{4}$; this point represents the transition from 4 distinct real-valued branch points to 4 complex-valued branch points (Figure 4.2 illustrates this). We also note that β is not monotonic in the parameter η; providing a physical explanation for this behavior is the subject of future work. Note that β specifies the exponential decay rate of the feedback k, this analysis formalizes the exponential decay rate observed in [40].

We recover the convolution kernel k from its Fourier transform $K_\lambda = K_\lambda = \begin{bmatrix} K_1(\lambda) & K_2(\lambda) \end{bmatrix}$ for the case of $\nu = 1$. We write

$$\begin{aligned} K_1(\lambda) &= \tilde{K}_1(\lambda) + K_1(\infty) := (K_1(\lambda) - 0.5) + 0.5, \\ K_2(\lambda) &= \tilde{K}_2(\lambda) + K_2(\infty) := (K_2(\lambda) - \sqrt{2}) + \sqrt{2}, \end{aligned}$$

so that $k_1(x)$ and $k_2(x)$ are given by

$$\begin{aligned} k_1(x) &= \tilde{k}_1(x) + 0.5 \cdot \delta(x), \\ k_2(x) &= \tilde{k}_2(x) + \sqrt{2} \cdot \delta(x), \end{aligned}$$

where δ is the Dirac delta distribution. The inverse Fourier transforms $\tilde{k}_1(x)$ and $\tilde{k}_2(x)$ of $\tilde{K}_1(\lambda)$ and $\tilde{K}_2(\lambda)$ are numerically computed and plotted in Figure 4.3 to illustrate the decay rate of the convolution kernels $\tilde{k}_1$ and $\tilde{k}_2$.

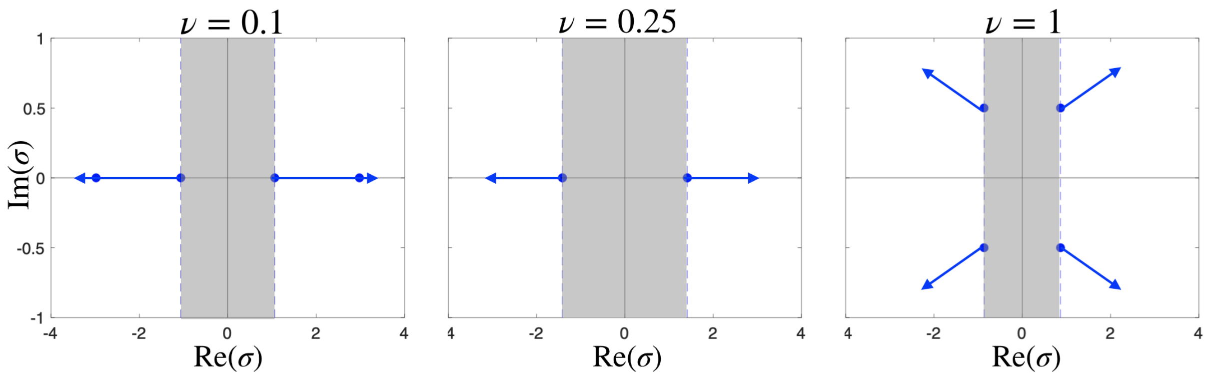


Figure 4.1: The blue lines denote the branch cuts for f_e for the case of $\nu = 0.1$ (left), $\nu = 0.25$ (center), and $\nu = 1$ (right). For $\nu = 0.1$ there are 4 real-valued branch points, for $\nu = 0.25$ there are 2 real-valued branch points, and for $\nu = 1$ there are 4 complex-valued branch points. The extension of the feedback, K_e, is analytic in the shaded region. The largest such region occurs for $\nu = 0.25$ and corresponds to branch cuts beginning at $\sqrt{2}$ on the Real axis.

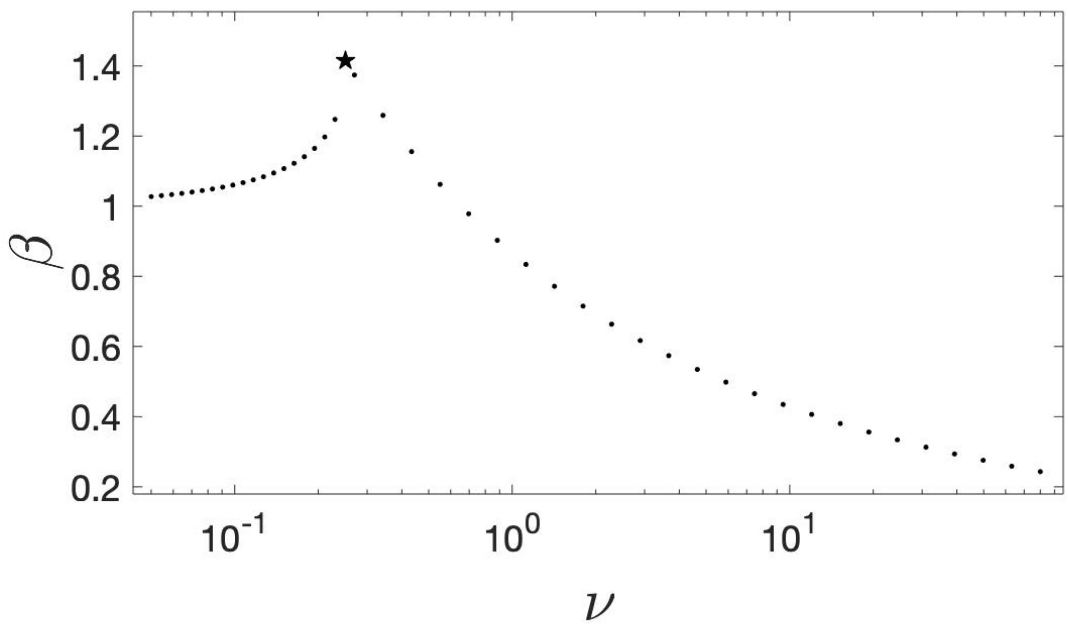


Figure 4.2: The boundary of the region of analyticity ($\beta = |\text{Re}(\sigma)|$) is plotted against the parameter $\nu = c^4\gamma$. The star denotes the non-differentiable point at ($\nu = \frac{1}{4}, \beta = \sqrt{2}$) which corresponds to the largest region of analyticity and thus the fastest decay rate. Note that the axis for ν is on a log scale.

4.6 Equivalence of L_2 & Sobolev Space Formulations

To analytically compute the decay rate of the optimal feedback k in Section 4.5, we explicitly computed the branch points of the function K_e. In this section, we provide an alternate method

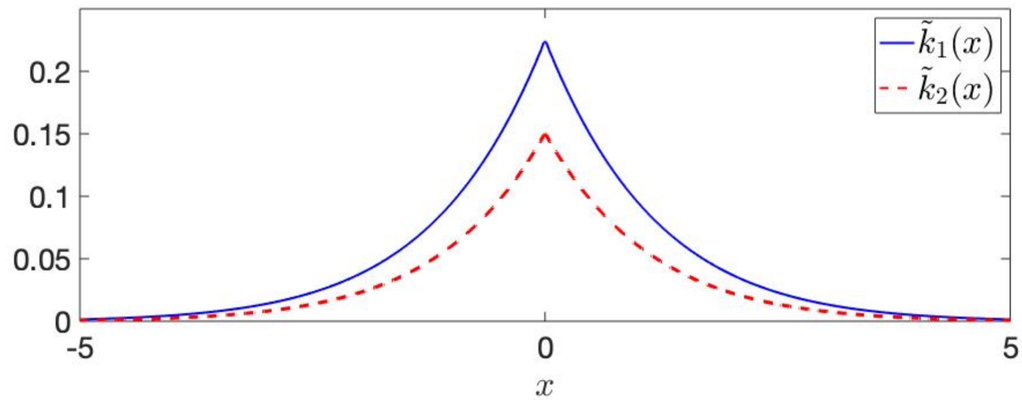


Figure 4.3: The decay rates of the convolution kernels, $\tilde{k}_1$ and $\tilde{k}_2$, are represented for the case $\nu = 1$. These were computed by numerical integration of the inverse Fourier transform formula of $\tilde{K}_1 = K_1(\lambda) - K_1(\infty)$ and $\tilde{K}_2 = K_2(\lambda) - K_2(\infty)$. The steady state terms $K_1(\infty)$ and $K_2(\infty)$ represent Dirac δ distributions in the convolution kernels k_1 and k_2 and were subtracted off before numerical integration.

to proving this exponential decay rate that is more generalizable as it avoids the need for these branch point computations. For clarity, we examine the wave equation with the same output that was analyzed in Section 4.5 as a concrete example throughout this section.

Consider the LQR problem (4.20) where A generates a C_0-semigroup on a standard Sobolev space $\mathcal{X} = \mathcal{H}^\alpha(\mathbb{R})$ for some mulitindex α, $B \in \mathcal{L}(L^2, \mathcal{X})$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ is bounded, and $R \in \mathcal{L}(L^2)$ is positive definite. Assume:

1. The LQR problem (4.20) is well-posed,
2. A, B, C, and R are spatially-invariant operators, and $\hat{a}_e, \hat{b}_e, \hat{c}_e$, and $\hat{r}_e$, are given by analytic, rational functions on some strip Γ_β of the complex plane.

Under these assumptions we will demonstrate the following:

- The LQR problem (4.20) over the *standard* Sobolev space $\mathcal{H}^\alpha(\mathbb{R})$ can be formulated as an equivalent LQR problem over L_2, and this reformulation is well-posed (Thm. 4.6.1) if the original problem is
- The optimal feedback for this reformulation decays exponentially; the optimal feedback for the original Sobolev space formulation decays with the same rate (Thm 4.6.2).

As an illustrative example, we first demonstrate that the LQR problem for the wave equation analyzed in Section 4.5 can be reformulated over an L_2 space.

Example 4.6.1 *An alternate state space representation of the wave equation dynamics is given by*

$$\begin{aligned}\partial_t \hat{\phi}(\lambda, t) &= \begin{bmatrix} 0 & 1 - i\lambda \\ \frac{-c^2\lambda^2}{1-i\lambda} & 0 \end{bmatrix} \hat{\phi}(\lambda, t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \hat{u}(\lambda, t) \\ &=: \hat{a}'_\lambda \hat{\phi}(\lambda, t) + \hat{b}'_\lambda \hat{u}(\lambda, t),\end{aligned} \tag{4.54}$$

where we have defined a new state variable by $\hat{\phi}_\lambda := \hat{s}_\lambda \hat{\psi}_\lambda$*, with*

$$W_\lambda = \begin{bmatrix} 1+i\lambda & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1-i\lambda & 0 \\ 0 & 1 \end{bmatrix} =: (\hat{s}_\lambda)^* \hat{s}_\lambda$$

a spectral factorization of the weighting matrix W*.* $M_{\hat{a}'}$ *generates a* C_0*-semigroup on* L_2*. Note that* $A' = \mathcal{F}^{-1} M_{\hat{a}'} \mathcal{F}$ *is not a differential operator. It is easily checked that* (A', B') *is exponentially stabilizable. The LQR problem* (4.47) *can be written in terms of this new state variable by writing the cost of state as*

$$\left\langle \hat{\phi}, M_{\hat{q}'} \hat{\phi} \right\rangle_{L_2}$$

where $\hat{q}'_\lambda := \hat{s}_\lambda \hat{q}_\lambda (\hat{s}^\dagger)_\lambda = \hat{s}_\lambda I (W_\lambda^{-1} \hat{s}^*_\lambda) = I$*.*

The following theorem generalizes the results of this example to any problem over a Sobolev space.

Theorem 4.6.1 *Consider the LQR problem* (4.20) *over a Sobolev space* $\mathcal{H}^\alpha(\mathbb{R})$ *with corresponding spatial frequency weighting matrix* W *and assume that (1), (2) hold. Let* $W_\lambda = \hat{s}^*_\lambda \hat{s}_\lambda$*, denote a spectral factorization of the weighting matrix* W_λ*. Then* (4.47) *can be formulated over an* L_2 *space as*

$$\begin{aligned} &\inf_u \int_{t=0}^{\infty} \left\langle C'\phi(t), C'\phi(t) \right\rangle_{L_2} + \langle u(t), Ru(t) \rangle_{L_2} \, dt \\ &s.t. \quad \partial_t \phi(t) = A'\phi(t) + B'u(t). \end{aligned} \tag{4.55}$$

where the state ϕ *is defined in the frequency domain as*

$$\hat{\phi}(\lambda, t) := \hat{s}_\lambda \hat{\psi}(\lambda, t) \tag{4.56}$$

and A', B' *and* C' *are computed in the frequency domain as*

$$\hat{a}'_\lambda := \hat{s}_\lambda \hat{a}_\lambda \hat{s}_\lambda^{-1}, \; \hat{b}'_\lambda := \hat{s}_\lambda \hat{b}_\lambda, \; \hat{c}'_\lambda = \hat{c}_\lambda \hat{s}_\lambda^{-1}. \tag{4.57}$$

This formulation (4.55) *is well-posed.*

Proof: It is straightforward to confirm that the state transformation (4.56) results in (4.55) where A', B', C' are defined in (4.57). It remains to check that well-posedness of the original LQR problem implies well-posedness of the transformed problem. First note that if (C, A) is detectible, then by Theorem 4.4.5 there exists a solution $\hat{p}_\lambda = \hat{p}^*_\lambda \succ 0$ of the matrix Riccati equation

$$\hat{p}_\lambda \hat{a}^*_\lambda + \hat{a}_\lambda \hat{p}_\lambda + W_\lambda^{-1} - \hat{p}_\lambda \hat{c}^*_\lambda V_\lambda \hat{c}_\lambda \hat{p}_\lambda = 0 \tag{4.58}$$

for a.e. $\lambda \in \mathbb{R} \backslash \{0\}$ and this solution satisfies the boundedness condition (4.46). Equation (4.58) can be written in terms of the transformed system parameters as

$$\begin{aligned} 0 &= (\hat{s}_\lambda \hat{p}_\lambda \hat{s}^*_\lambda)(\hat{a}')^* + \hat{a}'(\hat{s}_\lambda \hat{p}_\lambda \hat{s}^*_\lambda) + I - (\hat{s}_\lambda \hat{p}_\lambda \hat{s}^*_\lambda)(\hat{c}')^* V_\lambda \hat{c}'_\lambda (\hat{s}_\lambda \hat{p}_\lambda \hat{s}^*_\lambda) \\ &= \hat{\pi}_\lambda (\hat{a}')^* + \hat{a}' \hat{\pi}_\lambda + I - \hat{\pi}_\lambda (\hat{c}')^* V_\lambda \hat{c}'_\lambda \hat{\pi}_\lambda \end{aligned} \tag{4.59}$$

The operator $M_{\hat{\pi}}$ is self-adjoint and satisfies the uniform boundedness condition, so that by Theorem 4.4.5, the transformed system (C', A') is detectable. The proof that (A', B') is stabilizable follows similarly. Lastly, we demonstrate that boundedness of $e^{At}B$ implies boundedness of $e^{A't}B'$, where $\{e^{At}\}$ and $\{e^{A't}\}$ are the semigroups generated by A and A' respectively. These semigroups are related by the formula $e^{A't} = Se^{At}S^{-1}$. Then for any $u \in L_2$,

$$\begin{aligned} \|e^{A't}B'u\|_{L_2}^2 &= \|Se^{At}S^{-1}SBu\|_{L_2}^2 \\ &= \left\langle Se^{At}Bu, Se^{At}Bu \right\rangle_{L_2} \\ &= \left\langle e^{At}Bu, e^{At}Bu \right\rangle_H \\ &= \|e^{At}Bu\|_H^2, \end{aligned} \tag{4.60}$$

so that the norm of the operator $e^{A't}B'$ is the same as that of $e^{At}B$. ■

We next relate a decay rate of the solution of the transformed problem over an L_2 space to the decay rate of the solution to the original problem over a Sobolev space. We begin by looking at the wave equation example once again.

Example 4.6.2 *The optimal solution to problem* (4.55) *for the wave equation of Example 4.6.1 is given by*

$$\hat{u} = M_{\hat{k}'}\hat{\phi}\ ,\ \ \hat{k}'_\lambda := -\frac{1}{\gamma}(\hat{b}')^\dagger_\lambda \hat{p}'_\lambda, \tag{4.61}$$

where $M_{\hat{p}'}$ is a bounded self-adjoint operator and $\hat{p}'_\lambda = (\hat{p}')^\dagger_\lambda$ is the solution to the Riccati equation

$$(\hat{a}')^\dagger_\lambda \hat{p}'_\lambda + \hat{p}'_\lambda \hat{a}'_\lambda + \hat{q}'_\lambda = \frac{1}{\gamma}\hat{p}'_\lambda \hat{b}'_\lambda (\hat{b}')^\dagger_\lambda \hat{p}'_\lambda \tag{4.62}$$

for all $\lambda \in \mathbb{R}$. This problem is well-posed, so that (4.61) *is stabilizing. An extension of $\hat{a}'$ is given by the rational function*

$$\hat{a}'_e(\sigma) = \begin{bmatrix} 0 & 1-\sigma \\ \frac{c^2\sigma^2}{1-\sigma} & 0 \end{bmatrix}$$

which is analytic in Γ_1, and $\hat{q}'_e$ will be rational and analytic in some strip as well. Then, an application of [16, Thm 6] shows that $\hat{k}'_e$ (an extension of $\hat{k}'$) is analytic in Γ_η for some η and k' therefore decays exponentially with rate η. The feedback policies for both formulations are equivalent, i.e.

$$M_{\hat{k}}\hat{\psi} = M_{\hat{k}'}\hat{\phi}.$$

Moreover, from the relation

$$\hat{k}_e(\sigma) = \hat{k}'_e(\sigma) \cdot \begin{bmatrix} 1+\sigma & 0 \\ 0 & 1 \end{bmatrix}, \tag{4.63}$$

we see that $\hat{k}_e$ (an extension of $\hat{k}$) will be analytic in the same region Γ_η as $\hat{k}'_e$. Thus, k will have at least the same exponential decay rate as k'.

The following theorem generalizes the results of Example 4.6.2.

Theorem 4.6.2 *Let the optimal feedback for the transformed L_2 formulation of the LQR problem (4.55) be denoted by $u = k'\phi$ and the optimal feedback for the original Sobolev space formulation be denoted by $u = k\psi$. These two feedback policies are equivalent, i.e.*

$$K'\phi = K\psi,$$

and the decay rate of the convolution kernel k is at least as rapid as that of k'.

Proof: $\hat{k}_e(\sigma) = \hat{k}'_e(\sigma)\hat{s}_e(\sigma)$. As $\hat{s}_e$ (an extension of the spectral factor $\hat{s}$) is analytic, if $\hat{k}'_e$ is analytic in a given region Γ_β, then $\hat{k}_e$ will be as well. ■

We emphasize that the relation between the decay rates of the convolution kernels of optimal feedback for the original problem (k) and of the optimal feedback for the transformed problem (k') is *not* the same as the relation between the decay rates of the corresponding Riccati equation solutions p and p'. Assuming $\hat{r}_e^{-1}$ and $\hat{b}'_e$ are analytic, the branch points of

$$\hat{k}'_e = \hat{r}_e^{-1}(\hat{b}'_e)^*\hat{p}'_e \tag{4.64}$$

will be exactly the branch points of $\hat{p'}_e$. Thus, k' will have the same exponential decay rate as p'. In contrast, assuming $\hat{r}_e^{-1}$ and $\hat{b}_e$ are analytic, the branch points of

$$\hat{k}_e = \hat{r}_e^{-1}(\hat{b}_e)^* W_e \hat{p}_e, \tag{4.65}$$

will be exactly the branch points of $W_e\hat{p}_e$, *not* the branch points of $\hat{p}_e$. It can be shown that

$$W_e\hat{p}_e = \hat{s}_e^*\hat{p}'_e\hat{s}_e, \tag{4.66}$$

where $W_\lambda = \hat{s}_\lambda^*\hat{s}_\lambda$ is a spectral factorization. Since $\hat{s}_e$ and $\hat{s}_e^*$ are analytic, the branch points of $W_e\hat{p}_e$ are in fact the same as the branch points of $\hat{p}_e$. Moreover we see that the optimal feedback for the original Sobolev formulation K can be recovered from the solution $\hat{p}'$ of the Riccati equation for the transformed L_2 LQR problem:

$$\begin{aligned} \hat{k}_\lambda &= -\hat{r}_\lambda^{-1}\hat{b}_\lambda^* W_\lambda W_\lambda^{-1}\hat{s}_\lambda^*\hat{p}'_\lambda\hat{s}_\lambda \\ &= -\hat{r}_\lambda^{-1}\hat{b}_\lambda^*\hat{s}_\lambda^*\hat{p}'_\lambda\hat{s}_\lambda \end{aligned} \tag{4.67}$$

This analysis provides an alternate proof of Theorem 4.6.2.

4.7 Optimal Control of Wave Equation with Mechanical Energy Output

In this section, we analyze the optimal LQR controller design problem for the wave equation with mechanical energy output, which is related to a *homogeneous* Sobolev norm. A future line of work is in generalizing the results presented in this Section to analyze more general LQR problems for which the cost functional is described by a homogeneous Sobolev inner product.

We consider the following state space formulation of the wave equation

$$\partial_t\zeta(t) = \begin{bmatrix} 0 & I \\ c^2\partial_x^2 & 0 \end{bmatrix}\zeta(t) + \begin{bmatrix} 0 \\ \frac{1}{\rho}I \end{bmatrix}u(t) \quad =: \tilde{A}\zeta(t) + \tilde{B}u(t) \tag{4.68}$$

where $\mathcal{D}(\tilde{A})$ is dense in H and $\tilde{A} : \mathcal{D}(\tilde{A}) \to H$ generates a C_0-semigroup of bounded operators on the *homogeneous* Sobolev space $H = \mathcal{H}_0^1(\mathbb{R}) \oplus L_2(\mathbb{R})$ defined by (4.7), and $\tilde{B} \in \mathcal{L}(L^2(\mathbb{R}), H)$ is a bounded operator. The output is chosen so that the LQR cost functional captures a measure of mechanical energy:

$$E(t) := c^2 \|\partial_x \xi(\cdot, t)\|_{L_2}^2 + \|\partial_t \xi(\cdot, t)\|_{L_2}^2. \tag{4.69}$$

One such formulation is to define

$$y(t) = \begin{bmatrix} cI & 0 \\ 0 & I \end{bmatrix} \zeta(t), \tag{4.70}$$

and form the corresponding LQR problem

$$\begin{aligned} &\inf_k \int_0^\infty \langle y(t), y(t) \rangle_H + \langle u(t), \gamma u(t) \rangle_{L_2} \; dt \\ &\text{s.t. dynamics (4.68).} \end{aligned} \tag{4.71}$$

Note that with this choice $\langle y(t), y(t) \rangle_H = E(t)$. Direct calculations demonstrate that the LQR problem (4.71) is not well-posed. However, we reformulate this problem slightly to obtain a well-posed problem which optimizes an equivalent objective.

We consider the state space representation (4.13), for which the operator A generates a C_0-semigroup of bounded operators on the *standard* Sobolev space $\mathcal{X} = H^{(1,0)}(\mathbb{R})$:

$$\frac{d}{dt}\psi(t) = \begin{bmatrix} 0 & I \\ c^2 \partial_x^2 & 0 \end{bmatrix} \psi(t) + \begin{bmatrix} 0 \\ \frac{1}{\rho} I \end{bmatrix} u(t) =: A\psi(t) + Bu(t)$$

We choose an operator $C \in \mathcal{L}(\mathcal{X}, L^2)$ such that the output $y = C\psi$ provides a measure of mechanical energy at time t:

$$\langle y(t), y(t) \rangle_{L_2} = \left\langle \psi, C^\dagger C \psi \right\rangle_{\mathcal{H}^{(1,0)}} = E(t) \tag{4.72}$$

It is straightforward to check that this choice of C is given by

$$C = \begin{bmatrix} c\partial_x & 0 \\ 0 & I \end{bmatrix} \tag{4.73}$$

and that $C \in \mathcal{L}(\mathcal{H}^{(1,0)}(\mathbb{R}), L^2)$ is a bounded operator. The LQR problem

$$\begin{aligned} &\inf_k \int_0^\infty \langle C\psi(t), C\psi(t) \rangle_{L_2} + \langle u(t), \gamma u(t) \rangle_{L_2} \; dt \\ &\text{s.t. dynamics (4.13)} \end{aligned} \tag{4.74}$$

optimizes the same measure of performance as (4.71). To confirm that (4.74) is well-posed, we must check detectability of (C, A). To do so, we employ the following result.

Proposition 4.7.1 *Let A generate a C_0 semigroup of bounded operators on $\mathcal{X}$, $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ be a bounded operator, and define $Q := C^\dagger C \in \mathcal{L}(\mathcal{X})$. If (Q, A) is exponentially detectable, then (C, A) is exponentially detectable as well.*

Proof: Assume (Q, A) is detectable, or equivalently $(A^\dagger, Q^\dagger)$ is stabilizable. Then there exists a bounded operator $G \in \mathcal{L}(\mathcal{X})$ for which $(A^\dagger - Q^\dagger G) = (A^\dagger - QG)$ is exponentially stable, so that

$$\Pi(A^\dagger - QG) + (A - G^\dagger Q)\Pi + I = 0, \tag{4.75}$$

for some bounded positive definite operator $\Pi \in \mathcal{L}(\mathcal{X})$. Define $F := CG \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$. F is bounded because it is the composition of two bounded operators, and

$$\Pi(A^\dagger - C^\dagger F) + (A - F^\dagger C)\Pi + I \tag{4.76}$$

$$= \Pi(A^\dagger - C^\dagger CG) + (A - G^\dagger C^\dagger C)\Pi + I = 0, \tag{4.77}$$

so that (C, A) is detectable. ■

The symbol of the multiplication operator $M_{\hat{q}}$ with $Q = C^\dagger C$ with C defined by (4.73) is given by

$$\hat{q}_\lambda = \begin{bmatrix} \frac{c^2\lambda^2}{1+\lambda^2} & 0 \\ 0 & 1 \end{bmatrix}.$$

We confirm detectability of (Q, A) for A given by (4.13) using Theorem 4.4.5. The solution of the parameterized Riccati equation for detectability (4.45) of (Q, A) is given by $\hat{p}_\lambda = \begin{bmatrix} \frac{1}{c^2\lambda^2} & 0 \\ 0 & 1 \end{bmatrix}$ and the boundedness condition (4.46) is straightforward to verify.

The optimal feedback of LQR problem (4.74) is obtained using Theorem 4.4.6. The parameterized Riccati equation (4.49) for this problem is given by

$$\hat{p}_\lambda \begin{bmatrix} 0 & 1 \\ -c^2\lambda^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & -c^2\lambda^2 \\ 1 & 0 \end{bmatrix} \hat{p}_\lambda + \begin{bmatrix} 1+\lambda^2 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \frac{c^2\lambda^2}{1+\lambda^2} & 0 \\ 0 & 1 \end{bmatrix} - \frac{1}{\gamma^2}\hat{p}_\lambda \begin{bmatrix} 0 \\ \frac{1}{\rho} \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{\rho} \end{bmatrix} \hat{p}_\lambda$$

$$= \begin{bmatrix} -c^2\lambda^2(p_0 + p_0^*) + c^2\lambda^2 - \frac{1}{\rho^2\gamma^2}p_0 p_0^* & p_1 - c^2\lambda^2 p_2 - \frac{1}{\gamma^2\rho^2}p_0 p_2 \\ p_1 - c^2\lambda^2 p_2 - \frac{1}{\gamma^2\rho^2}p_0^* p_2 & p_0 + p_0^* + 1 - \frac{1}{\gamma^2\rho^2}p_2^2 \end{bmatrix} = 0 \tag{4.78}$$

We compute the solution $\hat{p}_\lambda = \begin{bmatrix} p_1 & p_0 \\ p_0 & p_2 \end{bmatrix} = \hat{p}_\lambda^*$ by solving the equation (4.78) entrywise. From the (1,2) and (2,1) entries we determine that $p_0 = p_0^*$ so that p_0 is real-valued. Then the (1,1) entry may be written as

$$\frac{1}{\rho^2\gamma^2}p_0^2 + 2c^2\lambda^2 p_0 - c^2\lambda^2 = 0, \tag{4.79}$$

and the remaining entries allow p_1 and p_2 to be written in terms of p_0 as

$$\begin{aligned} p_2 &= \frac{1}{\gamma^2\rho^2}\sqrt{1 + 2p_0} \\ p_1 &= \frac{1}{\gamma^2\rho^2}\sqrt{1 + 2p_0}\left(c^2\lambda^2 + \frac{1}{\gamma^2\rho^2}p_0\right) \end{aligned} \tag{4.80}$$

We compute a solution

$$\begin{aligned} \hat{p}_\lambda &:= \begin{bmatrix} c^2\lambda^2\rho\gamma\sqrt{2g(\lambda)+1}\sqrt{(\rho c\lambda\gamma)^2+1} & g(\lambda) \\ g(\lambda) & \rho\gamma\sqrt{2g(\lambda)+1} \end{bmatrix}, \\ g(\lambda) &:= -(\rho c\lambda\gamma)^2 + \sqrt{(\rho c\lambda\gamma)^4 + (\rho c\lambda\gamma)^2} \end{aligned} \tag{4.81}$$

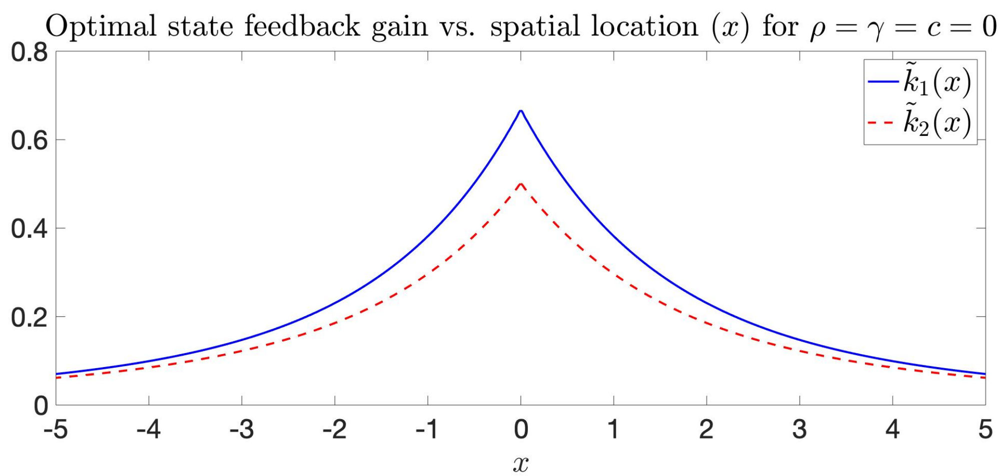


Figure 4.4: Convolution kernel k of optimal feedback for LQR problem (4.74) with parameters $c = \gamma = \rho = 1$.

for a.e. $\lambda \in \mathbb{R} \setminus \{0\}$. Note that each of the square roots is a multi-valued function, i.e. $\sqrt{r} = +r$ or $-r$, leading to multiple solutions $\hat{p}_\lambda$. We restrict to the positive choice of each square root function in (4.81) and confirm via the Schur complement test that with this choice (4.81) is the unique positive definite solution of (4.78).

The corresponding optimal feedback is

$$\begin{aligned} \hat{u}_\lambda(t) &= \hat{k}_\lambda \hat{\psi}_\lambda(t) \\ &= -\hat{r}_\lambda^{-1} \hat{b}_\lambda^* \hat{p}_\lambda \hat{\psi}_\lambda(t) \\ &= -\frac{1}{\gamma^2 \rho} \left[\begin{array}{cc} g(\lambda) & \rho\gamma\sqrt{2g(\lambda)+1} \end{array} \right] \hat{\psi}_\lambda(t), \end{aligned} \tag{4.82}$$

which results in the closed-loop dynamics

$$\begin{aligned} \frac{d}{dt}\hat{\psi}_\lambda(t) &= (\hat{a}_\lambda + \hat{b}_\lambda \hat{k}_\lambda)\hat{\psi}_\lambda(t) \\ &= \begin{bmatrix} 0 & 1 \\ -c^2\lambda^2 - \frac{1}{(\gamma\rho)^2} g(\lambda) & -\gamma\sqrt{2g(\lambda)+1} \end{bmatrix} \hat{\psi}_\lambda(t) \\ &= \begin{bmatrix} 0 & 1 \\ \frac{-c}{\rho\gamma}\sqrt{(\rho c\gamma)^2\lambda^4 + \lambda^2} & -\gamma\sqrt{2g(\lambda)+1} \end{bmatrix} \hat{\psi}_\lambda(t). \end{aligned} \tag{4.83}$$

In (4.82) and (4.83) we are restricting to the positive solution of each square root function. The convolution kernel of the optimal feedback is plotted in Figure 4.4 for the case of parameters $c = \rho = \gamma = 1$.

To prove the decay rate, we compute an extension $\hat{k}_e$ of $\hat{k}$

$$\hat{k}_e = \left[\begin{array}{cc} -\rho(c\sigma)^2 + \frac{c}{\gamma}(-i\sigma)\sqrt{(\rho\gamma c\sigma)^2 - 1} & -\frac{1}{\gamma}\sqrt{2g_e(\sigma) - 1} \end{array} \right] \tag{4.84}$$

where

$$g_e(\sigma) = (\rho c\gamma)^2\sigma^2 + \rho c\gamma(-i\sigma)\sqrt{(\rho c\gamma\sigma)^2 - 1}. \tag{4.85}$$

Note that (4.84) and (4.85) are both multivalued functions. The branch points of the multivalued function $l(\sigma) = \sqrt{(\rho c\gamma\sigma)^2 - 1}$ are given by

$$\sigma = \pm\frac{1}{\rho c\gamma}, \tag{4.86}$$

and we choose a corresponding branch cut that goes from $\frac{1}{\rho c\gamma}$ to ∞ and back to $-\frac{1}{\rho c\gamma}$, avoiding the imaginary axis. Thus, a branch of the function l_e is analytic in the strip

$$\left\{|\mathrm{Re}(\sigma)| < \frac{1}{\rho c\gamma}\right\}. \tag{4.87}$$

We need to also check for additional branch points of $\hat{k}_e$, which occur when $2g_e(\sigma) - 1 = 0$, i.e.$g_e(\sigma) = 1/2$. These values of σ are given by:

$$\sigma = \pm\frac{1}{\rho c\gamma}\sqrt{2 \pm \sqrt{2}}$$

which are outside of the region (4.87) so that we can choose a branch of $\hat{k}_e$ analytic in this strip, confirming exponential decay of the optimal feedback k. Note that the region of analyticity is inversely proportional to the wave speed c and the control cost parameters ρ, γ. In particular, the analyticity region (and thus the decay rate) increase with smaller wave speed c or smaller control cost $\gamma\rho$.

Generalizations: From Homogeneous to Standard Sobolev Space Cost

Recall that the results of Section 4.6 applied only to the setting of a *standard* Sobolev space as the state space. The analysis of Section 4.7 demonstrated that the LQR problem for the wave equation over a homogeneous Sobolev space with performance output measured by a homogeneous Sobolev norm could be transformed to an LQR problem over a standard Sobolev space which is well-posed. An interesting next question to ask is whether such a transformation exists more generally. Formally, consider an LQR problem for the system

$$\frac{d}{dt}\psi(t) = A\psi(t) + Bu(t), \tag{4.88}$$

where A generates a C_0-semigroup of bounded operators on the homogeneous Sobolev space $\mathcal{H}_0^\alpha(\mathbb{R})$ and $B : L_2(\mathbb{R}) \to \mathcal{H}_0^\alpha(\mathbb{R})$. The cost functional is of the form

$$\int_{t=0}^{\infty} \langle\psi(t), \psi(t)\rangle_{\mathcal{H}_0^\alpha(\mathbb{R})} + \langle u(t), u(t)\rangle_{L_2(\mathbb{R})}\ dt. \tag{4.89}$$

Assume that the dynamics can be written equivalently as

$$\frac{d}{dt}\psi(t) = \tilde{A}\psi(t) + \tilde{B}u(t), \tag{4.90}$$

where $\tilde{A}$ generates a C_0-semigroup of bounded operators on the *standard* Sobolev space $\mathcal{H}^\alpha(\mathbb{R})$ and $\tilde{B} : L_2(\mathbb{R}) \to \mathcal{H}^\alpha(\mathbb{R})$. The cost functional

$$\int_{t=0}^{\infty} \langle\psi(t), Q\psi(t)\rangle_{\mathcal{H}^\alpha(\mathbb{R})} + \langle u(t), u(t)\rangle_{L_2(\mathbb{R})}\ dt \tag{4.91}$$

where Q is defined in the spatial frequency domain as $\hat{q}_\lambda = W_\lambda \tilde{W}_\lambda^{-1}$, where W and $\tilde{W}$ are the spatial frequency weighting matrices of $\mathcal{H}_0^\alpha$ and $\mathcal{H}^\alpha$ is equivalent to the cost functional (4.89). Future work will try to determine whether (under certain assumptions) an LQR problem of the form (4.91) will be well-posed, thus allowing for methods of Section 4.6 to extend to the homogeneous Sobolev space setting.

4.8 Conclusion & Open Problems

We demonstrated that the optimal LQR feedback for a PDE over a standard Sobolev space is a spatial convolution operator with exponentially decaying kernel (under appropriate assumptions). This generalized the results of [16] which were presented for just an L_2 setting. The main tool we utilized was a transformation from a Sobolev space to a weighted L_2 space via the spatial frequency weighting matrix. As a first case study, we analyzed the LQR problem for the wave equation with a standard Sobolev norm for the cost functional. We computed the exponential decay rate of the optimal feedback for this problem as a function of the system parameters. This decay rate was observed to be a non-monotonic function of the system parameter of interest, and understanding this non-monotonic behavior is a subject of future work.

As a case study, we examined the LQR controller design problem to optimize a mechanical energy measure for the wave equation. The cost functional of this problem could be formulated as a homogeneous Sobolev norm, or as a quadratic form on a standard Sobolev space. With the standard Sobolev space formulation, the problem was shown to be well-posed, and the optimal feedback decayed exponentially in space. A future research direction is in proving whether this result holds more generally. Formally, when can a cost functional described by a homogeneous Sobolev norm be reformulated as quadratic form on a standard Sobolev space, with this new standard Sobolev space representation leading to a well-posed LQR problem?

Additional interesting and related open problems include imposing convex constraints on the decay rate of feedback to extend results of e.g. [22] to the continuous spatial domain setting. The following chapter works toward this line of research as well.

Chapter 5

An Operator Perspective of System Level Synthesis

***Abstract* - The System Level Synthesis (SLS) framework provides a method for controller design by providing an affine linear parameterization of all achievable stabilized closed-loop mappings for a system to be controlled. In this chapter we develop an operator framework for this SLS methodology and provide a parameterization of all achievable closed-loop mappings for systems over a general Banach space. This general framework recovers previous results for systems over a finite spatial domain and spatially-invariant systems over a countably infinite spatial domain, in both continuous and discrete-time settings. We demonstrate that this general framework also applies to the setting of control of PDEs. As a case study, we analyze the LQR controller design problem for the diffusion equation. For this example, we compare our closed-loop design methods with existing Riccati equation methods for controller design for PDEs.**

5.1 Background & Introduction

The optimal controller design problem subject to structural constraints remains an open problem in the most general settings. The System Level Synthesis (SLS) methodology [11] provides an alternate and computationally tractable approach to the structured controller design problem, by directly designing the *closed-loops* rather than the controller. As emphasized in [32, 33] and Chapter 3, the optimal *structured closed-loop* design problem analyzed by SLS is not the same as the optimal *structured controller* design problem. However, the optimal controller design problem subject to (convex) constraints on the closed-loops is convex, and the corresponding controller allows for an implementation which inherits this structure.

SLS was originally presented in the context of discrete-time finite-dimensional systems [11]. Throughout this dissertation, this closed-loop design procedure was shown to easily generalize to the continuous-time setting, and additional analysis provided an equivalent result for the spatially-invariant setting over a (countably) infinite spatial domain. A natural next question is whether analogous results hold in the setting of an *uncountable* spatial domain as well. Such an extension would allow an SLS like parameterization to be applied to the optimal controller design problem for PDEs.

It was shown in [16] that the optimal LQR feedback for an *unconstrained* controller design problem for a PDE over an L^2 space is a spatial convolution kernel with an inherent exponential decay rate. Chapter 4 provided similar results for the unconstrained LQR problem for PDEs over a Sobolev space. In contrast, the proposed SLS framework would allow for analysis of controller design problems for PDEs in which locality or structural constraints are explicitly imposed. Indeed, this constrained controller design problem for PDEs remains an open problem except in special cases, e.g. funnel causality [8].

An extension of the SLS methodology to the PDE setting would allow for a convex formulation of a structured closed-loop design problem for PDEs, and would recover a controller with a structured implementation. This could provide a first step toward the design of controllers with localized/ sparse actuation for applications in fluid dynamics [18, 19], biology, and various other fields.

Thus, in this Chapter we begin to consider such an extension of the SLS framework. Rather than focusing solely on the setting of PDEs though, we take a step back and consider a more general framework. We provide a closed-loop parameterization of all stabilizing controllers for general systems over Banach spaces. This parameterization allows an optimal controller design problem for a system over a Banach space to be written in terms of the closed-loop mappings. Convex structural constraints on the closed-loops may be imposed while preserving convexity of this optimization problem. The operator perspective we consider is quite general, and we demonstrate that special cases of this framework include finite-dimensional settings originally introduced in [11] as well as spatially-invariant settings over a discrete spatial domain (as considered in Chapter 2). To demonstrate that the usefulness of this general framework indeed extends beyond existing results, we apply it to the controller design problem for spatially-invariant systems over a continuous spatial domain (PDEs).

The remainder of this Chapter is structured as follows. In Section 5.2 we define stability of (continuous- or discrete-time) systems over a Banach space. In Section 5.3 we define notions of stability and well-posedness of interconnections of systems over Banach spaces. We provide a closed-loop parameterization of all achievable and stabilized closed-loop mappings for this class of systems in Section 5.4 and employ this parameterization to formulate an optimal controller design problem in Section 5.5. We specialize our results to the spatially-invariant setting in Section 5.6. In Section 5.7 we apply our methods to the LQR controller design problem for the diffusion equation over the real line.

5.2 Notation & Mathematical Preliminaries

We use calligraphic letters to denote Banach spaces, e.g. $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{U}$. We denote (continuous-time or discrete-time) signals which take values in a Banach space by lower-case letters, i.e.

$$\psi : \mathcal{T} \to \mathcal{X}, \tag{5.1}$$

where $\mathcal{T}$ denotes the time domain so that $\psi(t) \in \mathcal{X}$ is the value of the signal ψ at time $t \in \mathcal{T}$. $\mathcal{T} = \mathbb{R}^+ := [0, \infty)$ for continuous-time and $\mathcal{T} = \mathbb{Z}^+ = \{0, 1, ...\}$ for discrete-time settings. We allow for vector valued signals as well, i.e. $\psi(t) \in \mathcal{X}^n$ for each t; we omit this superscript to simplify notation, writing $\psi(t) \in \mathcal{X}^n$ simply as $\psi(t) \in \mathcal{X}$.

We introduce the following subsets of signals.

- The L^p space

$$L^p_{\mathcal{X}}(\mathcal{T}) := \left\{ \psi : \mathcal{T} \to \mathcal{X};\ \ \|\psi\|^p_{L^p_{\mathcal{X}}} := \int_{t \in T} \|\psi(t)\|^p_{\mathcal{X}} dt < \infty \right\}. \tag{5.2}$$

  Note that $L^p_{\mathcal{X}}(\mathcal{T})$ is a Banach space.

- The extended L^p space

$$L^{p,e}_{\mathcal{X}}(\mathcal{T}) := \left\{ \psi : \mathcal{T} \to \mathcal{X};\ \int_{\substack{t \in \mathcal{T}, \\ t < \infty}} \|\psi(t)\|^p_{\mathcal{X}} dt < \infty, \quad \text{for all } T < \infty \right\}. \tag{5.3}$$

  Note that $L^{p,e}_{\mathcal{X}}(\mathcal{T})$ is a vector space, but it is not normed.

We use $\mathcal{S}$ to denote either the temporal differentiation operator on continuous-time signals or the temporal shift operator on discrete time-signals. Given a $\mathcal{X}$-valued signal ψ over $\mathcal{T}$,

$$(\mathcal{S}\psi)(t) := \begin{cases} \frac{d}{dt}\psi(t), & \mathcal{T} = \mathbb{R}^+ \\ \psi(t+1), & \mathcal{T} = \mathbb{Z}^+ \end{cases} \tag{5.4}$$

Given two Banach spaces $\mathcal{U}$ and $\mathcal{X}$, $\mathcal{L}(\mathcal{U}, \mathcal{X})$ denotes the space of linear operators from $\mathcal{U}$ to $\mathcal{X}$; to simplify notation we write $\mathcal{L}(\mathcal{X}) = \mathcal{L}(\mathcal{X}, \mathcal{X})$. An operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ is *bounded* if

$$\|B\|_{\mathcal{U} \to \mathcal{X}} := \sup_{\|u\|_{\mathcal{U}=1}} \|Bu\|_{\mathcal{X}} < \infty.$$

The domain of a linear operator $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ is denoted by $\mathcal{D}(B) \subset \mathcal{U}$, and we assume that all unbounded operators are defined on dense domains of the underlying space, i.e. $\mathcal{D}(B)$ is dense in $\mathcal{U}$.

Let $A \in \mathcal{L}(\mathcal{X})$ denote a (possibly unbounded) operator on the Banach space $\mathcal{X}$ with dense domain $\mathcal{D}(A) \subset \mathcal{X}$. Assume:

$$A \text{ generates a (strongly continuous) } C_0\text{-semigroup } \{T(t)\}_{t \in \mathcal{T}} \text{ on } \mathcal{X} \tag{5.5}$$

and consider the corresponding abstract difference or differential equation

$$\mathcal{S}\psi(t) = A\psi(t), \quad \psi(0) = \psi_0 \in \mathcal{D}(A) \subset \mathcal{X}. \tag{5.6}$$

The unique solution of (5.6) is given by

$$\psi(t) = T(t)\psi_0.$$

In the continuous-time setting, we often denote $T(t) = e^{At}$. We consider a continuous or discrete-time system G with dynamics

$$\mathcal{S}\psi(t) = A\psi(t) + Bu(t) \tag{5.7a}$$

$$y(t) = C\psi(t) + Du(t), \tag{5.7b}$$

where $\psi(t) \in \mathcal{X}$ for each t. For each t, $u(t)$ and $y(t)$ are elements of the Banach spaces $\mathcal{U}$ and $\mathcal{Y}$ respectively. Again we allow for vector valued signals, i.e. $\psi(t) \in \mathcal{X}^n, u(t) \in \mathcal{U}^m, y(t) \in \mathcal{Y}^p$, but omit these superscripts to simplify notation. We assume:

$$D \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \text{ is a bounded operator} \tag{5.8}$$

and $B \in \mathcal{L}(\mathcal{U}, \mathcal{X})$ and $C \in \mathcal{L}(\mathcal{X}, \mathcal{Y})$ are (possibly unbounded) operators but are such that

$$CT(t)B \in \mathcal{L}(\mathcal{U}, \mathcal{Y}) \text{ is bounded for each } t \in \mathcal{T}, \text{ and } \{CT(t)B\}_{t\in\mathcal{T}} \text{ is strongly continuous in } t. \tag{5.9}$$

We refer to (5.7a) as the state equation and (5.7b) as the output equation.

Definition 5.2.1 *The system G* (5.7) *is said to be* strictly proper *if the D operator of its realization* (5.7) *is zero.*

Given an input signal $u : \mathcal{T} \to \mathcal{U}$, and assuming zero initial condition, the dynamics of the system G described by (5.7) are given formally by

$$\begin{aligned} y(t) &= \int_{\tau\in\mathcal{T},\ t\leq T} CT(\tau)Bu(t-\tau)d\tau \ + \ Du(t) \\ &=: \int_{\tau\in\mathcal{T},\ t\leq T} G(\tau)u(t-\tau)d\tau, \end{aligned} \tag{5.10}$$

For each t, $G(t) \in \mathcal{L}(\mathcal{U}, \mathcal{Y})$, and we refer to this operator-valued sequence $\{G(t)\}$ as the *impulse response* of G. Note that the integral (5.10) is taken over a Banach space, and is well-defined if it converges absolutely, i.e.

$$\int_{\tau\in\mathcal{T},\ t\leq T} \|G(t)u(t-\tau)\|_{\mathcal{Y}} d\tau < \infty, \tag{5.11}$$

which occurs when $u \in L^{p,e}_{\mathcal{U}}(\mathcal{T})$ and G satisfies the assumptions (5.5), (5.8) and (5.9).

Definition 5.2.2 *The system G with impulse response $G(t)$ is* bounded-input bounded output (BIBO) stable *if*

$$\|G\|_{1-i} = \|G\|_{\infty-i} := \int_{t\in\mathcal{T}} \|G(t)\|_{\mathcal{U}\to\mathcal{Y}} \ dt < \infty. \tag{5.12}$$

Note that by the Reisz Convexity Theorem,

$$\|G\|_{p-i} \leq \|G\|_{\infty-i}, \quad \text{for all } 1 \leq p \leq \infty, \tag{5.13}$$

and thus BIBO stability implies L^p stability for all $1 \leq p \leq \infty$,.

5.3 Stability of Feedback Interconnections

We consider the controller design problem for systems over Banach spaces. In particular, the plant we consider is of the form

$$\mathcal{S}\psi(t) = A\psi(t) + B_1 d(t) + B_2 u(t) \tag{5.14a}$$

$$\overline{z}(t) = C_1\psi(t) + D_{11}d(t) + D_{12}u(t) \tag{5.14b}$$

$$y(t) = C_2\psi(t) + D_{21}d(t) + D_{22}u(t), \tag{5.14c}$$

where $u(t), \psi(t), y(t), \overline{z}(t)$ are in the Banach spaces $\mathcal{U}, \mathcal{X}, \mathcal{Y}$, and $\mathcal{Z}$ respectively, for each $t \in \mathcal{T}$. We assume that the system with state equation (5.14a) and output equations (5.14b) and (5.14c) satisfy assumptions (5.5), (5.8) and (5.9). We make the additional assumption that $D_{11} = 0$ and $D_{22} = 0$ and that $C_1 \in \mathcal{L}(\mathcal{X}, \mathcal{Z})$ is a bounded operator.

The output feedback controller K to be designed for plant (5.14) is of the form

$$\begin{aligned} \mathcal{S}\xi(t) &= A_c\xi(t) + B_c y(t) \\ u(t) &= C_c\xi(t) + D_c y(t). \end{aligned} \tag{5.15}$$

A_c generates a C_0 semigroup of bounded operators $\{T_c(t)\}_{t\in\mathcal{T}}$ on $\mathcal{X}_c$, $C_cT_c(t)B_c \in \mathcal{L}(\mathcal{Y}_c, \mathcal{U}_c)$ is a bounded operator for each $t \in \mathcal{T}$, and $D_c \in \mathcal{L}(\mathcal{Y}, \mathcal{U})$ is a bounded operator.

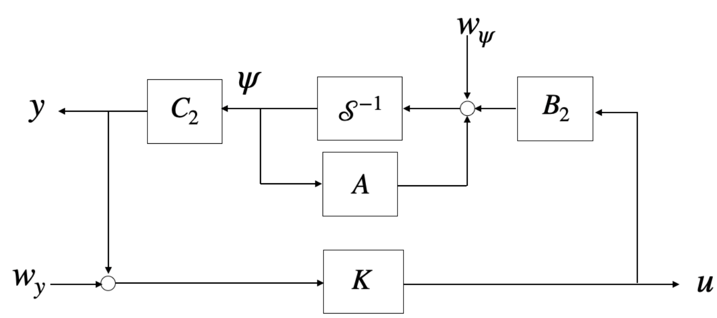


Figure 5.1: Feedback interconnection of plant P (5.14) with output feedback controller K (**??**). w_ψ is the exogenous disturbance entering into the state equation and w_y is the exogenous disturbance entering into the output equation

The feedback interconnection of plant P (5.14) with controller K (5.15) (see Figure 5.1) is described by the following relations:

$$\begin{bmatrix} (\mathcal{S} - A - B_2KC_2) & 0 \\ 0 & (\mathcal{S} - A - KC_2B_2) \end{bmatrix} \begin{bmatrix} \psi \\ u \end{bmatrix} = \begin{bmatrix} I & B_2K \\ KC_2 & KC_2B_2 + (\mathcal{S} - A - KC_2B_2)K \end{bmatrix} \begin{bmatrix} w_\psi \\ w_y \end{bmatrix}. \tag{5.16}$$

When there exists a unique solution $(\psi, u) \in L^{p,e}_{\mathcal{X}}(\mathcal{T}) \times L^{p,e}_{u}(\mathcal{T})$ for each pair $(w_\psi, w_y) \in L^{p,e}_{\mathcal{X}}(\mathcal{T}) \times L^{p,e}_{\mathcal{Y}}(\mathcal{T})$, the interconnection is said to be *well-posed*. When the interconnection is well-posed, operators on the left hand side of (5.16) are invertible over the $L^{p,e}$ spaces, and this relation may be written as

$$\begin{aligned}\begin{bmatrix} \psi \\ u \end{bmatrix} &= \begin{bmatrix} (\mathcal{S}-A-B_2KC_2)^{-1} & (\mathcal{S}-A-B_2KC_2)^{-1}B_2K \\ (\mathcal{S}-A-KC_2B_2)^{-1}KC_2 & (\mathcal{S}-A-KC_2B_2)^{-1}KC_2B_2+K \end{bmatrix} \begin{bmatrix} w_\psi \\ w_y \end{bmatrix} \\ &=: \begin{bmatrix} \Phi^{\psi\psi} & \Phi^{\psi y} \\ \Phi^{u\psi} & \Phi^{uy} \end{bmatrix} \begin{bmatrix} w_\psi \\ w_y \end{bmatrix}\end{aligned} \tag{5.17}$$

We refer to the mappings

$$\Phi = \begin{bmatrix} \Phi^{\psi\psi} & \Phi^{\psi y} \\ \Phi^{u\psi} & \Phi^{uy} \end{bmatrix}$$

as the *closed-loops* of the feedback interconnection. The performance output $\overline{z}$ can be written in terms of these closed-loops as

$$\begin{aligned}\overline{z} &= \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^{\psi\psi} & \Phi^{\psi y} \\ \Phi^{u\psi} & \Phi^{uy} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} d \\ &=: \mathcal{F}(P;K)\, d\end{aligned} \tag{5.18}$$

Definition 5.3.1 *The output feedback controller K (5.15) is* internally stabilizing *for plant P (5.14) if the closed-loop mappings from w_ψ and w_y to state ψ and control u (see Figure 5.1) are each stable according to Definition 5.2.2.*

5.3.1 State Feedback

The set-up presented in this section simplifies in the setting of state feedback control. We assume access to state measurement $y = \psi$, i.e. $C_2 = I, D_{21} = 0$.

In this state feedback setting, the interconnection of P with K simplifies to

$$\begin{bmatrix} \mathcal{S}-A-B_2K & 0 \\ 0 & \mathcal{S}-A-KB_2 \end{bmatrix} \begin{bmatrix} \psi \\ u \end{bmatrix} = \begin{bmatrix} I & B_2K \\ K & KB_2+(\mathcal{S}-A-KB_2)K \end{bmatrix} \begin{bmatrix} w_\psi \\ w_y \end{bmatrix} \tag{5.19}$$

which can be written (when well-posed) as

$$\begin{aligned}\begin{bmatrix} \psi \\ u \end{bmatrix} &= \begin{bmatrix} (\mathcal{S}-A-B_2K)^{-1} & (\mathcal{S}-A-B_2K)^{-1}B_2K \\ (\mathcal{S}-A-KB_2)^{-1}K & (\mathcal{S}-A-KB_2)^{-1}KB_2+K \end{bmatrix} \begin{bmatrix} w_\psi \\ w_y \end{bmatrix} \\ &=: \begin{bmatrix} \Phi^{\psi} & H^{\psi} \\ \Phi^{u} & H^{u} \end{bmatrix} \begin{bmatrix} w_\psi \\ w_y \end{bmatrix}\end{aligned} \tag{5.20}$$

In the case that $w_y = 0$, this reduces further to

$$\begin{aligned}\begin{bmatrix} \psi \\ u \end{bmatrix} &= \begin{bmatrix} I \\ K \end{bmatrix} (\mathcal{S}-A-B_2K)^{-1} w_\psi \\ &= \begin{bmatrix} \Phi^{\psi} \\ \Phi^{u} \end{bmatrix} w_\psi\end{aligned} \tag{5.21}$$

and we refer to $\Phi = \begin{bmatrix} \Phi^{\psi} \\ \Phi^{u} \end{bmatrix}$ as the closed-loops. To simplify notation, we often write w_ψ simply as w. The mappings H^u and H^ψ can be written in terms of these closed-loops as

$$\begin{aligned}H^{\psi} &= \Phi^{\psi}(\mathcal{S}-A) - I \\ H^{u} &= \Phi^{u}(\mathcal{S}-A).\end{aligned} \tag{5.22}$$

From realizations (5.15) and (5.14), we recover the following (non-unique) realizations of the closed-loops (5.21) resulting from this feedback interconnection (when $w_y = 0$) :

$$\begin{aligned}\zeta(t) &:= \begin{bmatrix} \psi(t) \\ \xi(t) \end{bmatrix} \\ \mathcal{S}\zeta(t) &= \begin{bmatrix} A + BD_c & BC_c \\ B_c & A_c \end{bmatrix} \zeta(t) + \begin{bmatrix} I \\ 0 \end{bmatrix} w(t) \\ \psi(t) &= \begin{bmatrix} I & 0 \end{bmatrix} \zeta(t) \\ u(t) &= \begin{bmatrix} D_c & C_c \end{bmatrix} \zeta(t)\end{aligned} \tag{5.23}$$

Note that (5.23) demonstrates that Φ^ψ an Φ^u (5.21) are strictly proper.

Thus, a check for stability in the state feedback setting is provided by the following proposition.

Proposition 5.3.1 *The state feedback controller K (5.15) is* internally stabilizing *for plant P (5.14) iff the following four mappings are stable:*

$$\begin{aligned}\Phi^\psi &:= (\mathcal{S} - A - B_2K)^{-1} \\ \Phi^u &:= K(\mathcal{S} - A - B_2K)^{-1} \\ H^\psi &:= \Phi^\psi(\mathcal{S} - A) - I \\ H^u &:= \Phi^u(\mathcal{S} - A).\end{aligned} \tag{5.24}$$

When A is a bounded operator, stability of Φ^ψ and Φ^u imply stability of H^ψ and H^u. In this setting Proposition 5.3.1 simplifies as follows.

Corollary 5.3.2 *Assume A is a bounded operator. Then the state feedback controller K (5.15) is internally stabilizing if and only if Φ^ψ and Φ^u are stable.*

5.4 Closed-Loop Parameterizations

In this section, we provide a parameterization of all achievable closed-loop mappings for plant (5.14) in terms of the resulting closed-loops defined by (5.17). For simplicity of exposition, throughout the remainder of this chapter we often present results in only the state feedback setting, noting that analogous results for the output feedback setting follow similarly.

Theorem 5.4.1

1. *There exists a state feedback controller K (5.15) for plant P (5.14) that results in the closed-loops (5.21) if and only if these closed-loops Φ^ψ and Φ^u are strictly proper and satisfy the affine condition*

$$(\mathcal{S} - A)\Phi^\psi - B_2\Phi^u = I, \tag{5.25}$$

where I is the identity operator on $L^{p,e}_{\mathcal{X}}(\mathcal{T})$.

2. *The corresponding controller can be recovered from the closed-loops as*

$$K = \Phi^u(\Phi^\psi)^{-1}. \tag{5.26}$$

Proof: First, let Φ^ψ and Φ^u be the closed-loops resulting from a feedback interconnection of P with some state feedback controller K. Then, from (5.21) Φ^ψ and Φ^u satisfy the affine relation (5.51) and realization (5.23) shows Φ^ψ and Φ^u are strictly proper.

Conversely, assume Φ^ψ and Φ^u are strictly proper and satisfy (5.51) and define

$$K := \Phi^u(\Phi^\psi)^{-1} \tag{5.27}$$

Then the closed-loop mapping from w to control u is given by

$$\begin{aligned} K(\mathcal{S} - A - B_2K)^{-1} &= \Phi^u(\Phi^\psi)^{-1}\left(\mathcal{S} - A - B_2\Phi^u(\Phi^\psi)^{-1}\right)^{-1} \\ &= \Phi^u\left((\mathcal{S} - A)\Phi^\psi - B_2\Phi^u\right)^{-1} \qquad = \Phi^u, \end{aligned} \tag{5.28}$$

and the closed-loop mapping from w to ψ is given by

$$\begin{aligned} (\mathcal{S} - A - B_2K)^{-1} &= \left(\mathcal{S} - A - B_2\Phi^u(\Phi^\psi)^{-1}\right)^{-1} \\ &= \left((\mathcal{S} - A)\Phi^\psi - B_2\Phi^u\right)\Phi^\psi \qquad = \Phi^\psi. \end{aligned} \tag{5.29}$$

■

Note that the parameterization of all achievable stabilized closed-loops for finite dimensional discrete-time systems provided in [11, Thm.2] is a special case of this result. The parameterization of all achievable stabilized closed-loops for spatially-invariant (discrete- or continuous-time) systems over a countable spatial domain, presented in Chapter 2, is a special case of this result as well.

5.5 Optimal Controller Design

We design the controller K (5.15) to internally stabilize the plant P (5.14) and optimize some norm of the mapping from disturbance d to performance output $\overline{z}$ in closed-loop. Formally, the optimal controller design problem of interest is of the form:

$$\begin{aligned} &\inf_K \quad \|\mathcal{F}(P;K)\| \\ &\text{s.t.} \quad K \text{ internally stabilizing} \\ = \quad &\inf_K \quad \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^{\psi\psi} & \Phi^{\psi y} \\ \Phi^{u\psi} & \Phi^{uy} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \right\| \\ &\text{s.t.} \quad K \text{ internally stabilizing} \end{aligned} \tag{5.30}$$

for some appropriately defined norm $\|\cdot\|$. This problem simplifies in the state feedback setting to

$$\begin{aligned} &\inf_K \quad \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^{\psi} \\ \Phi^{u} \end{bmatrix} B_1 \right\| \\ &\text{s.t.} \quad K \text{ internally stabilizing} \end{aligned} \tag{5.31}$$

The state feedback optimal controller design problem (5.31) can be written in terms of the closed-loops, as stated in the following corollary of Theorem 5.4.1.

Corollary 5.5.1 *The optimal controller design problem* (5.31) *can be written equivalently as*

$$\begin{array}{ll} \inf_{\Phi^\psi, \Phi^u} & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^\psi \\ \Phi^u \end{bmatrix} B_1 \right\| \\ s.t. & \Phi^\psi, \Phi^u, \text{ stable \& strictly proper,} \\ & (\mathcal{S} - A)\Phi^\psi - B_2\Phi^u = I, \\ & H^\psi, H^u \text{ stable.} \end{array} \tag{5.32}$$

When A is a bounded operator, this reduces further to

$$\begin{array}{ll} \inf_{\Phi^\psi, \Phi^u} & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^\psi \\ \Phi^u \end{bmatrix} B_1 \right\| \\ s.t. & \Phi^\psi, \Phi^u \text{ stable \& strictly proper,} \\ & (\mathcal{S} - A)\Phi^\psi - B_2\Phi^u = I. \end{array} \tag{5.33}$$

5.6 Diagonalizable & Spatially-Invariant Systems

It is often the case that computations of stability and performance of systems simplify when the system is *diagonalizable*. We present a review of diagonalizable systems in this section, with spatially-invariant systems analyzed as a special case. In this setting, we show that the closed-loop parameterizations and optimal controller design problems simplify in this setting.

Definition 5.6.1 *Given a set $\hat{\Omega}$, let $(\mathbb{C}^n)^{\hat{\Omega}}$ denote the set of all vector-valued functions $f : \hat{\Omega} \to \mathbb{C}^n$. Given a matrix valued-function $\hat{a} : \hat{\Omega} \to \mathbb{C}^{n\times m}$, the operator $M_{\hat{a}} : (\mathbb{C}^m)^{\hat{\Omega}} \to (\mathbb{C}^n)^{\Omega}$ defined by*

$$(M_{\hat{a}}v)(\lambda) := \hat{a}(\lambda)v(\lambda), \quad \lambda \in \hat{\Omega} \tag{5.34}$$

is called the multiplication operator *associated with the function $\hat{a}$. To simplify notation, we often write $\hat{a}_\lambda = \hat{a}(\lambda)$.*

Definition 5.6.2 *A linear operator $A : \mathcal{W} \to \mathcal{W}$ on a normed linear space $\mathcal{W}$ is* diagonalizable *if there exists a set $\hat{\Omega}$, a function $\hat{a} : \hat{\Omega} \to \mathbb{C}$, a function space $\hat{\mathcal{W}} \subset \{f : \hat{\Omega} \to \mathbb{C}\}$ equipped with a norm $\|\cdot\|_{\hat{\mathcal{W}}}$ and an invertible transformation $V : \hat{\mathcal{W}} \to \mathcal{W}$ that converts A into the multiplication operator*

$$V^{-1}AV = M_{\hat{a}}. \tag{5.35}$$

Definition 5.6.3 *The spatio-temporal system G* (5.7) *is* diagonalizable *if each of the operators A, B, C, D is diagonalizable by a single transformation V.*

The diagonalized system, denoted by $\hat{G}$, is of the form

$$\begin{aligned} \mathcal{S}\hat{\psi}(t) &= M_{\hat{a}}\hat{\psi}(t) + M_{\hat{b}}\hat{u}(t) \\ \hat{y}(t) &= M_{\hat{c}}\hat{\psi}(t) + M_{\hat{d}}\hat{u}(t), \end{aligned} \tag{5.36}$$

where $\hat{\psi}(t), \hat{u}(t)$, and $\hat{y}(t) \in \hat{\mathcal{X}}, \hat{\mathcal{U}}$ and $\hat{\mathcal{Y}}$, respectively. We let $\hat{T}(t)$ denote the C_0-semigroup generated by $M_{\hat{a}}$. Note that these dynamics (5.36) are *decoupled* in $\lambda \in \hat{\Omega}$ and can be written pointwise in λ as

$$\begin{aligned} \mathcal{S}\hat{\psi}_\lambda(t) &= \hat{a}\hat{\psi}_\lambda(t) + \hat{b}_\lambda\hat{u}_\lambda(t) \\ \hat{y}_\lambda(t) &= \hat{c}_\lambda\hat{\psi}_\lambda(t) + \hat{d}_\lambda\hat{u}_\lambda(t), \end{aligned} \tag{5.37}$$

For each λ, we let $\hat{G}_\lambda$ denote the finite dimensional system (5.37).

In certain settings checking stability of the diagonalization of a system may be easier than checking stability of the original system. Note that the diagonalizing transformation creates a new norm on the space $\mathcal{W}$: for $\psi \in \mathcal{W}$ define

$$\|\psi\|_V := \|V\psi\|_{\hat{\mathcal{W}}}. \tag{5.38}$$

Assume a diagonalizing transformation V is such that this new norm is equivalent to the original norm on $\mathcal{W}$ in the sense that there exist constants α_1, α_2 for which

$$\alpha_1\|\psi\|_{\mathcal{W}} \leq \|\psi\|_V \leq \alpha_2\|\psi\|_{\mathcal{W}} \tag{5.39}$$

for all $\psi \in \mathcal{W}$. Then a system G (which is diagonalizable by V) is stable if and only if the transformed system $\hat{G} = VGV^{-1}$ is.

Spatially-Invariant Systems

Spatially-invariant systems are one subclass of diagonalizable systems. In order for this notion of spatial-invariance to be well-defined, we assume that the underlying Banach spaces describe spatial signals over some spatial domain Ω that is a (possibly uncountable) set equipped with a measure and a notion of addition. In particular, we assume that Ω is a (locally compact) commutative group. Given such a set Ω we define the spatial signal space

$$\mathbb{C}^\Omega := \{f : \Omega \to \mathbb{C}\}. \tag{5.40}$$

One important subset of this function space is the set of square integrable functions:

$$L^2(\Omega) := \left\{ f : \Omega \to \mathbb{C}^n;\ \|f\|^2_{L^2(\Omega)} := \int_{\lambda\in\Omega} f^*(\lambda)f(\lambda)d\mu(\lambda) < \infty \right\}, \tag{5.41}$$

where μ denotes a measure on Ω.

Definition 5.6.4 *Let $\Omega = \mathbb{G}$ denote a (locally compact) Abelian group, and for each y in $\mathbb{G}$ define the shift operator T_y on $\mathbb{C}^{\mathbb{G}}$ by*

$$(T_y f)(x) := f(x - y).$$

An operator A on $L_2(\mathbb{G})$ is said to be spatially-invariant *if it commutes with all such shift operators, i.e.*

$$T_y A = AT_y, \quad \text{for all } y \in \mathbb{G}.$$

One type of spatially-invariant operator is a spatial convolution operator, formally defined as follows.

Definition 5.6.5 *An operator $B \in \mathcal{L}(\mathbb{C}^{\mathbb{G}})$ of the form*

$$Bu(x, t) = \int_{\xi\in\mathbb{G}} b(x - \xi)u(\xi)d\mu(\xi) \tag{5.42}$$

is a spatial convolution operator*. b is referred to as the* convolution kernel *of B*

Definition 5.6.6 *Let G be a system of the form* (5.7) *with state, input and output spaces $\mathcal{X}, \mathcal{U}, \mathcal{Y} \subset \mathbb{C}^{\mathbb{G}}$. Then G is* spatially-invariant *if each operator A, B, C, D of its state space representation are spatial convolution operators.*

Note that the input, output, and state of a spatially-invariant system are each *spatio-temporal signals*, e.g.

$$(\psi(t))(x) = \psi(x,t), \quad x \in \mathbb{G},\ t \in \mathcal{T}$$

When the plant of interest is spatially invariant, we restrict our attention to controllers which are as well.

Proposition 5.6.1 *A controller K for a spatially-invariant plant P is itself spatially-invariant if and only if the corresponding closed-loops Φ are.*

The diagonalizing transform for spatially-invariant systems is the Fourier transform. The general Fourier transform is defined on (locally compact) commutative groups $\mathbb{G}$ as follows. Let $f : \mathbb{G} \to \mathbb{C}$, then

$$\hat{f}(\lambda) = (\mathcal{F}f)(\lambda) := \int_{x\in\mathbb{G}} f(x) e^{-i\lambda x} dx, \quad \lambda \in \hat{\mathbb{G}}, \tag{5.43}$$

where $\hat{\mathbb{G}}$ is the *dual group* of $\mathbb{G}$. The dual groups of $\mathbb{R}$, $\mathbb{T}$, $\mathbb{Z}$, $\mathbb{Z}_N$ are summarized in the following table.

Group ($\mathbb{G}$)	$\mathbb{R}$	$\mathbb{T}$	$\mathbb{Z}$	$\mathbb{Z}_N$
Dual group ($\hat{\mathbb{G}}$)	$\mathbb{R}$	$\mathbb{Z}$	$\mathbb{T}$	$\mathbb{Z}_N$

$\mathcal{F} : L_2(\mathbb{G}) \to L_2(\hat{\mathbb{G}})$ is an isometric isomorphism,

$$\langle f, g \rangle_{L_2(\mathbb{G})} = \langle \mathcal{F}f, \mathcal{F}g \rangle_{L_2(\hat{\mathbb{G}})} \tag{5.44}$$

and diagonalizes any spatial convolution operator (5.42), i.e.

$$M_{\hat{b}} \hat{u}_\lambda(t) := \hat{b}_\lambda(t) \hat{u}_\lambda(t), \quad \lambda \in \hat{\mathbb{G}}, \tag{5.45}$$

where $\hat{b} = \mathcal{F}b$, the Fourier transform of the convolution kernel b of B, $M_{\hat{b}} = \mathcal{F}B\mathcal{F}^{-1}$ and $\hat{u}_\lambda(t) = (\mathcal{F}u)(\lambda, t) = \int_{x\in\mathbb{G}} u(x,t) e^{-ix\lambda} dx$.

Since (5.44) implies condition (5.39), stability of a spatially-invariant system over L^2 is equivalent to stability of its diagonalization.

Stability of Spatially-Invariant Systems over L^2

In this L^2 setting, the $\mathcal{H}_\infty$ norm provides a measure of stability.

The $\mathcal{H}_\infty$ norm of a finite dimensional system H is defined as

$$\|H\|_{\mathcal{H}_\infty} := \sup_{u \in L^2[0,\infty)} \frac{\int_{t\in\mathcal{T}} (Hu)^*(t)(Hu)(t)dt}{\int_{t\in\mathcal{T}} u^*(t)u(t)dt}, \tag{5.46}$$

and can be interpreted as the worst case 'energy amplification' from input to output. The computation of the $\mathcal{H}_\infty$ norm of a finite-dimensional transfer function is simplified in the case that the transfer function is block diagonal. Let $H(s)$ be a finite-dimensional transfer function block partitioned as

$$\begin{bmatrix} Y_1(s) \\ \vdots \\ Y_n(s) \end{bmatrix} = \begin{bmatrix} H_1(s) & & \\ & \ddots & \\ & & H_n(s) \end{bmatrix} \begin{bmatrix} U_1(s) \\ \vdots \\ U_n(s) \end{bmatrix}, \tag{5.47}$$

$\|H\|_{\mathcal{H}_\infty}$ can be computed from each of the block components of H as

$$\|H\|_{\mathcal{H}_\infty} = \max_{1\le k\le n} \|H_k\|_{\mathcal{H}_\infty} = \max_{1\le k\le n} \operatorname*{ess\,sup}_{\omega\in\mathbb{R}} \sigma_{\max}\{H_k(i\omega)\} \tag{5.48}$$

In the discrete-time setting,

$$\|H\|_{\mathcal{H}_\infty} = \max_{1\le k\le n} \|H_k\|_{\mathcal{H}_\infty} = \max_{1\le k\le n} \operatorname*{ess\,sup}_{\theta\in[0,2\pi)} \sigma_{\max}\{H_k(e^{i\theta})\} \tag{5.49}$$

We generalize this idea to define the $\mathcal{H}_\infty$ measure to the class of spatially-invariant systems.

Definition 5.6.7 *Let G be a spatially-invariant and let $\hat{G}$ denote its diagonalization. The $\mathcal{H}_\infty$ norm of G is given by*

$$\begin{aligned} \|G\|_{\mathcal{H}_\infty} &= \|\hat{G}\|_{\mathcal{H}_\infty} \\ &:= \operatorname*{ess\,sup}_{\lambda\in\hat{\Omega}} \|\hat{G}_\lambda\|_{\mathcal{H}_\infty} \\ &= \begin{cases} \operatorname*{ess\,sup}_{\omega\in\mathbb{R},\ \lambda\in\hat{\Omega}} \sigma_{\max}\{\hat{G}_\lambda(i\omega)\}, & \mathcal{T} = \mathbb{R}^+ \\ \operatorname*{ess\,sup}_{\theta\in[0,2\pi),\ \lambda\in\hat{\Omega}} \sigma_{\max}\{\hat{G}_\lambda(e^{i\theta})\}, & \mathcal{T} = \mathbb{Z}^+ \end{cases} \end{aligned} \tag{5.50}$$

The spatially-invariant system G is *stable* if it has finite $\mathcal{H}_\infty$ norm.

Note that stability can be checked pointwise in spatial frequency in the case that the spatial frequency domain $\hat{\mathbb{G}}$ is compact. In this case $\|G\|_{\mathcal{H}_\infty}$ is finite if and only if $\|\hat{G}_\lambda\|_{\mathcal{H}_\infty} < \infty$ for all $\lambda \in \hat{\mathbb{G}}$, i.e. if and only if each $\hat{G}_\lambda$ is stable.

Note that the preceding analysis holds for other classes of diagonalizable systems as well. Indeed the only assumptions on the diagonalizing transformation that we utilized were that the equivalent norm condition (5.39) holds and that the norm of the transformed system *decouples*, i.e.$\|\hat{H}\| = \int_{\lambda\in\hat{\Omega}} \|\hat{H}_\lambda\|$.

5.6.1 Closed-Loop Parameterizations for Spatially-Invariant Systems

The closed-loop parameterization provided in Theorem 5.4.1 'decouples' in the spatially-invariant setting as follows.

Corollary 5.6.2 *Let P be a spatially-invariant plant over a spatial domain $\mathbb{G}$. Then*

1. *There exists a spatially-invariant state feedback controller K for P that results in the spatially-invariant closed-loops Φ^ψ and Φ^u (5.21) if and only if these closed-loops are strictly proper and satisfy the parameterized family of affine conditions:*

$$(sI - \hat{a}_\lambda)\hat{\Phi}^\psi_\lambda(s) - (\hat{b}_2)_\lambda \hat{\Phi}^u_\lambda(s) = I, \text{ for all } \lambda \in \hat{\Omega}. \tag{5.51}$$

2. *The corresponding controller K is internally stabilizing if $\Phi^\psi, \Phi^u, H^\psi$, and H^u each have finite $\mathcal{H}_\infty$ norm, i.e.*

$$\begin{aligned} &\operatorname*{ess\,sup}_{\lambda\in\hat{\Omega}} \|\hat{\Phi^\psi}_\lambda\|_{\mathcal{H}_\infty} < \infty \\ &\operatorname*{ess\,sup}_{\lambda\in\hat{\Omega}} \|\hat{\Phi^u}_\lambda\|_{\mathcal{H}_\infty} < \infty \\ &\operatorname*{ess\,sup}_{\lambda\in\hat{\Omega}} \|\hat{H^\psi}_\lambda\|_{\mathcal{H}_\infty} < \infty \\ &\operatorname*{ess\,sup}_{\lambda\in\hat{\Omega}} \|\hat{H^u}_\lambda\|_{\mathcal{H}_\infty} < \infty \end{aligned} \tag{5.52}$$

3. *K can be recovered from the closed-loops in the spatial frequency domain as*

$$\hat{k}_\lambda(s) = \hat{\Phi}^u_\lambda(s)(\hat{\Phi}^\psi_\lambda(s))^{-1} \tag{5.53}$$

Note that in the discrete-time setting, the parameter s is replaced by z in Equations 5.51 and 5.53 of Corollary 5.6.2.

Condition (2) of Corollary 5.6.2 can be further simplified in the case that the spatial domain $\mathbb{G}$ is discrete, i.e. $\mathbb{G} = \mathbb{Z}_N$ or $\mathbb{Z}$. In this case, the operator A is necessarily bounded so that formulation (5.33) holds. In addition, for this choice of $\mathbb{G}$ discrete, the dual group $\hat{\mathbb{G}}$ is compact, allowing for stability to be checked pointwise in spatial frequency.

Corollary 5.6.3 *Let P be a spatially-invariant plant over a* discrete *spatial domain $\mathbb{G}$. Then a spatially-invariant state feedback controller K internally stabilizes P if and only if the resulting closed-loops satisfy*

$$\hat{\Phi}^\psi_\lambda, \hat{\Phi}^u_\lambda \;\; \text{stable for each } \lambda \in \hat{\mathbb{G}},$$

where stability is checked with the $\mathcal{H}_\infty$ norm.

$\mathcal{H}_2$ Controller Design for Spatially-Invariant Systems

Recall that the $\mathcal{H}_2$ norm of a transfer function H with m inputs and p outputs is computed as

$$\|H\|^2_{\mathcal{H}_2} = \int_{\omega\in\mathbb{R}} \operatorname{Tr}\left(H^*(i\omega)\, H(i\omega)\right) d\omega, \tag{5.54}$$

and provides a measure of the energy of the impulse response of H, i.e.

$$\|H\|^2_{\mathcal{H}_2} = \sum_{i=1}^{p}\sum_{j=1}^{m} \int_{t\in\mathcal{T}} \|h_{ij}(t)\|_2^2 dt. \tag{5.55}$$

This simplifies in the case that H is a spatially-invariant (circulant) transfer function. In this case, the $\mathcal{H}_2$ norm provides a (scaled) measure of the L_2 norm of the output of H subject to an impulse input at a spatial location k, i.e. $w(t) = \delta(t)e_k$, with e_k the k^{th} standard unit basis vector. Due to spatial invariance, the choice of k is arbitrary. Using this idea, we define an $\mathcal{H}_2$ measure of a diagonalizable system over a potentially infinite dimensional spatial domain.

Definition 5.6.8 *Let G be a spatially-invariant system over the spatial domain Ω and let $\hat{G}$ denote its diagonalization. The $\mathcal{H}_2$ norm of G is defined by*

$$\|G\|_{\mathcal{H}_2}^2 = \|M_{\hat{G}}\|_{\mathcal{H}_2}^2 := \int_{\lambda\in\hat{\Omega}} \|\hat{G}_\lambda\|_{\mathcal{H}_2}^2 d\lambda, \tag{5.56}$$

where $\|\hat{G}_\lambda\|_{\mathcal{H}_2}^2$ is the $\mathcal{H}_2$ norm of the finite dimensional transfer function $\hat{G}_\lambda$.

The optimal spatially-invariant $\mathcal{H}_2$ controller design problem for a spatially-invariant plant P is formally stated as follows.

$$\begin{array}{rll} & \inf\limits_{K)} & \|\mathcal{F}(P;K)\|_{\mathcal{H}_2}^2 \\ & \text{s.t.} & K \text{ internally stabilizing \& spatially-invariant} \\ = & \inf\limits_{K} & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^{\psi\psi} & \Phi^{\psi y} \\ \Phi^{u\psi} & \Phi^{uy} \end{bmatrix} \begin{bmatrix} B_1 \\ D_{21} \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \\ & \text{s.t.} & K \text{ internally stabilizing \& spatially-invariant} \end{array} \tag{5.57}$$

In the state feedback setting (5.57) reduces to

$$\begin{array}{rll} & \inf\limits_{K} & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^{\psi} \\ \Phi^{u} \end{bmatrix} B_1 \right\|_{\mathcal{H}_2}^2 \\ & \text{s.t.} & K \text{ internally stabilizing \& spatially-invariant} \\ & \inf\limits_{\Phi^\psi,\Phi^u} & \left\| \begin{bmatrix} C_1 & D_{12} \end{bmatrix} \begin{bmatrix} \Phi^{\psi} \\ \Phi^{u} \end{bmatrix} B_1 \right\|_{\mathcal{H}_2}^2 \\ = & \text{s.t.} & \Phi^\psi, \Phi^u, \text{ stable, strictly proper, \& spatially-invariant} \\ & & (\mathcal{S} - A)\Phi^\psi - B_2\Phi^u = I, \\ & & H^\psi, H^u \text{ stable.} \end{array} \tag{5.58}$$

The optimal spatially-invariant state feedback $\mathcal{H}_2$ controller design problem (5.58) can be written equivalently as

$$\begin{array}{ll} \inf\limits_{\Phi^\psi,\Phi^u} & \left\| \begin{bmatrix} M_{\hat{c}_1} & M_{\hat{d}_{12}} \end{bmatrix} \begin{bmatrix} \hat{\Phi}^{\psi} \\ \hat{\Phi}^{u} \end{bmatrix} M_{\hat{b}_2} \right\| \\ \text{s.t.} & \hat{\Phi}^\psi_\lambda, \hat{\Phi}^u_\lambda, \text{ strictly proper for each } \lambda \in \Omega \\ & (sI - \hat{a}_\lambda)\hat{\Phi}^\psi_\lambda(s) - (\hat{b}_2)_\lambda \hat{\Phi}^u_\lambda(s) = I, \text{ for all } \lambda \in \hat{\Omega}, \\ & \mathcal{H}_\infty \text{ stability condition (5.52)} \end{array} \tag{5.59}$$

In the case of a discrete spatial domain Ω, this reduces further to

$$\begin{array}{ll} \inf\limits_{\Phi^\psi,\Phi^u} & \left\| \begin{bmatrix} M_{\hat{c}_1} & M_{\hat{d}_{12}} \end{bmatrix} \begin{bmatrix} \hat{\Phi}^{\psi} \\ \hat{\Phi}^{u} \end{bmatrix} M_{\hat{b}_2} \right\| \\ \text{s.t.} & \hat{\Phi}^\psi_\lambda, \hat{\Phi}^u_\lambda \text{ stable and strictly proper for all } \lambda \in \hat{\Omega} \\ & (sI - \hat{a}_\lambda)\hat{\Phi}^\psi_\lambda(s) - (\hat{b}_2)_\lambda \hat{\Phi}^u_\lambda(s) = I, \text{ for all } \lambda \in \hat{\Omega}. \end{array} \tag{5.60}$$

The diagonalized problem (5.59) decouples into a family of optimization problems parameterized by $\lambda \in \hat{\mathbb{G}}$, each with finitely many transfer function parameters.

Remark 6 *In Chapter 4, we highlighted that Sobolev spaces can be identified with (weighted) L^2 spaces. From this perspective we see that the results of this Section can be easily generalized from the L^2 space setting to the Sobolev space setting.*

5.7 Application: Control of PDEs

In this Section, we specialize the results of the previous section to analyze the controller design problem for PDEs. We begin by analyzing a specific example - optimal LQR control of the diffusion equation.

5.7.1 Example: Diffusion

We consider the diffusion equation over the real line with fully distributed control u and disturbance w, described by the PDE

$$\partial_t \psi(x,t) = \alpha \partial_x^2 \psi(x,t) + w(x,t) + u(x,t), \quad x \in \mathbb{G} = \mathbb{R}, t \in \mathcal{T} = \mathbb{R}^+. \tag{5.61}$$

We design a state feedback controller for (5.61) to optimize an LQR objective. The dynamics (5.61) and performance output for this problem can be written in the form (5.14) as

$$\begin{aligned} \frac{d}{dt}\psi(t) &= A\psi(t) + w(t) + u(t) \\ \overline{z}(t) &= \begin{bmatrix} qI \\ 0 \end{bmatrix} \psi(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} u(t) \\ &=: C_1\psi(t) + D_{12}u(t) \end{aligned} \tag{5.62}$$

where the operator $A = \alpha \partial_x^2$ is defined on the domain $\mathcal{D}(A) = \{f \in L^2(\mathbb{R}) : f'' \in L^2(\mathbb{R})\} \subset L_2(\mathbb{R})$ dense in $L_2(\mathbb{R})$, and generates a C_0-semigroup $\{e^{At}\}$ on $L_2(\mathbb{R})$. The parameter q determines the relative cost of state to cost of control. We note that system (5.62) is *spatially-invaraint* so that the spatial Fourier transform provides a diagonalizing transformation. We design a spatially-invariant state feedback controller K for this system; equivalently, we restrict the resulting closed-loops to be spatially-invariant. The corresponding optimal controller design problem is a standard (infinite dimensional) LQR problem

$$\begin{array}{cl} \inf\limits_{K \text{ stabilizing}} & J = \int_0^\infty \left\langle \psi(t), q^2 I \psi(t) \right\rangle_{L^2} + \langle u(t), u(t) \rangle_{L^2} \, dt, \\ \text{s.t.} & \frac{d}{dt}\psi(t) = A\psi(t) + Iu(t), \quad \psi(0) = \psi_0 \\ & K \text{ spatially-invariant} \end{array} \tag{5.63}$$

(5.63) can be written in the form (5.58) as

$$\begin{array}{cl} \inf\limits_{\Phi^\psi, \Phi^u} & \left\| \begin{bmatrix} qI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \Phi^\psi \\ \Phi^u \end{bmatrix} I \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} & \Phi^\psi, \Phi^u \text{ stable, strictly proper, spatially-invariant} \\ & (\mathcal{S} - \alpha \partial_x)\Phi^\psi - \Phi^u = I \\ & H^\psi, H^u \text{ stable,} \end{array} \tag{5.64}$$

where

$$\begin{aligned} H^{\psi}(s) &:= \Phi^{\psi}(s)(sI - \alpha\partial_x^2) - I, \\ H^{u}(s) &:= \Phi^{u}(s)(sI - \alpha\partial_x^2). \end{aligned} \tag{5.65}$$

Note that the spatial domain $\mathbb{R}$ is non compact and the operator $A = \alpha\partial_x$ is unbounded so that (5.64) can not be reduced to the form (5.60), but a spatial Fourier transform does allow (5.64) to be written in the decoupled form (5.59). To convert (5.64) to this decoupled form, we write the dynamics (5.61) as

$$\frac{d}{dt}\hat{\psi}_\lambda(t) = \hat{a}_\lambda\hat{\psi}_\lambda(t) + \hat{u}_\lambda(t) + \hat{w}_\lambda(t), \quad \lambda \in \hat{\Omega} = \mathbb{R},\ t \in \mathcal{T} = \mathbb{R}^+ \tag{5.66}$$

where $M_{\hat{a}} = \mathcal{F}A\mathcal{F}^{-1}$ is the multiplication operator with symbol $\hat{a} := -\alpha\lambda^2$. Then (5.64) is equivalent to

$$\begin{aligned} \inf_{\hat{\Phi}^{\psi},\hat{\Phi}^{u}} \quad & \left\| \begin{bmatrix} qI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\Phi}^{\psi} \\ \hat{\Phi}^{u} \end{bmatrix} \right\|_{\mathcal{H}_2}^2 \\ \text{s.t.} \quad & \hat{\Phi}^{\psi}_\lambda, \hat{\Phi}^{u}_\lambda \text{ strictly proper for each } \lambda \in \mathbb{R}, \\ & (sI + \alpha\lambda^2)\hat{\Phi}^{\psi}_\lambda(s) - \hat{\Phi}^{u}_\lambda(s) = I \text{ for each } \lambda \in \mathbb{R}, \\ & \mathcal{H}_\infty \text{ stability condition (5.52)}, \end{aligned} \tag{5.67}$$

where the objective decouples as

$$\left\| \begin{bmatrix} qI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\Phi}^{\psi} \\ \hat{\Phi}^{u} \end{bmatrix} \right\|_{\mathcal{H}_2}^2 = \int_{\lambda\in\mathbb{R}} \left\| \begin{bmatrix} qI & 0 \\ 0 & I \end{bmatrix} \begin{bmatrix} \hat{\Phi}^{\psi}_\lambda \\ \hat{\Phi}^{u}_\lambda \end{bmatrix} \right\|_{\mathcal{H}_2}^2 d\lambda. \tag{5.68}$$

Motivated by the results presented in Chapter 2, we note that the affine constraint of (5.67) can be written as

$$\begin{bmatrix} (s+1) - (-\alpha\lambda^2 + 1) & -1 \end{bmatrix} \begin{bmatrix} \hat{\Phi^{\psi}}_\lambda(s) \\ \hat{\Phi^{u}}_\lambda(s) \end{bmatrix} = 1, \text{ for all } \lambda \in \mathbb{R}. \tag{5.69}$$

The following *explicit* parameterization of admissible $\hat{\Phi^{\psi}}_\lambda$ and $\hat{\Phi^{u}}_\lambda$ follows from (5.69).

Lemma 5.7.1 *For a fixed $\lambda \in \hat{\Omega} = \mathbb{R}$, the (static or dynamic) state feedback controller $\hat{K}_\lambda(s)$ is internally stabilizing for the finite dimensional system* (5.66) *iff the corresponding closed-loop mappings $\hat{\Phi}^{\psi}_\lambda, \hat{\Phi}^{u}_\lambda$ are of the form*

$$\begin{aligned} \hat{\Phi^{\psi}}_\lambda(s) &= \frac{1}{s+1}\left(1 + \hat{\rho}_\lambda(s)\right), \\ \hat{\Phi^{u}}_\lambda(s) &= \frac{\alpha\lambda^2 - 1}{s+1} + \frac{s + \alpha\lambda^2}{s+1}\hat{\rho}_\lambda(s), \end{aligned} \tag{5.70}$$

for some strictly proper transfer function $\hat{\rho}_\lambda(s)$ with all poles in the open left half plane.

The proof of Lemma 5.7.1 is provided in the Appendix. This result allows (5.67) to be solved via the following family of optimization problems, parameterized in $\lambda \in \mathbb{R}$.

$$\begin{aligned} &\inf_{\hat{\rho}_\lambda} \quad q^2 \left\| \frac{1}{s+1}\left(1+\hat{\rho}_\lambda(s)\right) \right\|_{\mathcal{H}_2}^2 + \left\| \frac{\alpha\lambda^2 - 1}{s+1} + \frac{s+\alpha\lambda^2}{s+1}\hat{\rho}_\lambda(s) \right\|_{\mathcal{H}_2}^2 \\ &\text{s.t.} \quad \hat{\rho}_\lambda(s) \text{ stable and strictly proper} \\ = &\inf_{\hat{\rho}_\lambda} \quad \|H_\lambda + V_\lambda \hat{\rho}_\lambda(s)\|_{\mathcal{H}_2}^2 \\ &\text{s.t.} \quad \hat{\rho}_\lambda(s) \text{ stable and strictly proper,} \end{aligned} \tag{5.71}$$

where

$$H_\lambda := \begin{bmatrix} \frac{q}{s+1} \\ \frac{\alpha\lambda^2 - 1}{s+1} \end{bmatrix}, \quad V_\lambda := \begin{bmatrix} \frac{q}{s+1} \\ \frac{s+\alpha\lambda^2}{s+1} \end{bmatrix}.$$

For each fixed $\lambda \in \mathbb{R}$, (5.71) is in the form of a standard model matching problem [31] with a single transfer function parameter; the optimal solution $\hat{\rho}_\lambda^{\text{opt}}(s)$ of this problem is given by

$$\hat{\rho}_\lambda^{\text{opt}}(s) = V_o^{-1} \cdot \left(V_i^{-1} H\right)\Big|_{RH_\infty},$$

where $V_\lambda = V_i V_o$ is an inner outer factorization and $\left(V_i^{-1} H\right)\Big|_{RH_\infty}$ is the projection of $V_i^{-1} H$ onto the subspace of stable real-rational strictly proper transfer functions. We compute:

$$\hat{\rho}_\lambda^{\text{opt}}(s) = \frac{1 - \sqrt{q^2+\gamma^2}}{s + \sqrt{q^2+\gamma^2}}, \tag{5.72}$$

where we have defined the parameter $\gamma := \alpha\lambda^2$. The details of this computation are provided in the Appendix. The optimal closed-loop mappings are recovered from the optimal parameter $\hat{\rho}_\lambda^{\text{opt}}(s)$ as:

$$\begin{aligned} \hat{\Phi^\psi}_\lambda(s) &= \frac{1+\hat{\rho}_\lambda(s)}{s+1} &&= \frac{1}{s+\sqrt{q^2+\gamma^2}} \\ \hat{\Phi^u}_\lambda(s) &= \frac{(\gamma-1)+(s+\gamma)\hat{\rho}_\lambda(s)}{s+1} &&= \frac{\gamma - \sqrt{q^2+\gamma^2}}{s+\sqrt{q^2+\gamma^2}} \end{aligned} \tag{5.73}$$

From (5.73), we compute

$$\begin{aligned} \hat{K}_\lambda^{\text{opt}}(s) = \hat{\Phi^u}_\lambda \hat{\Phi^\psi}_\lambda^{-1} &= \gamma - \sqrt{q^2+\gamma^2} \\ &= \alpha\lambda^2 - \sqrt{q^2+\alpha^2\lambda^4}. \end{aligned} \tag{5.74}$$

The solution to the $\mathcal{H}_2$ design problem for the diagonalized system (5.67) is given by the multiplication operator with symbol k_λ in the spatial Fourier representation. To verify whether this controller is internally stabilizing, we must confirm the condition (5.52) - details of these computations are provided in the Appendix.

Comparison to Standard LQR Approach

The LQR controller design problem for this system (5.63) can also be solved using the Riccati equation based techniques of [16]. We write this problem in the spatial frequency domain as

$$\begin{aligned} &\inf_{K \text{ stabilizing}} \quad J = \int_0^\infty \left\langle \psi(t), q^2 I \psi(t) \right\rangle_{L^2} + \langle u(t), u(t) \rangle_{L^2} \, dt, \\ &\quad \text{s.t.} \quad \frac{d}{dt}\psi(t) = a\psi(t) + bu(t), \quad \psi(0) = \psi_0 \\ = &\inf_{\hat{K} \text{ stabilizing}} \quad J = \int_{\lambda \in \mathbb{R}} \int_0^\infty q^2 \hat{\psi}_\lambda^2(t) + \hat{u}_\lambda^2(t) \; dt d\lambda \\ &\quad \text{s.t.} \quad \frac{d}{dt}\hat{\psi}_\lambda(t) = -\alpha\lambda^2 \hat{\psi}_\lambda(t) + \hat{u}_\lambda(t), \quad \hat{\psi}_\lambda(0) = \hat{\psi}_{\lambda,0}, \quad \lambda \in \mathbb{R} \end{aligned} \tag{5.75}$$

The cost and constraint decouple in spatial frequency parameter $\lambda \in \mathbb{R}$, allowing this problem to be solved via a family of finite dimensional LQR problems parameterized in $\lambda \in \mathbb{R}$. The solution of each such parameterized problem is obtained via a solution to the Riccati equation:

$$-\alpha\lambda^2 p_\lambda - p_\lambda \alpha \lambda^2 - p_\lambda p_\lambda + q^2 = 0 \quad \Rightarrow \quad p_\lambda = -\alpha\lambda^2 + \sqrt{\alpha^2\lambda^4 + q^2}, \tag{5.76}$$

resulting in the optimal static state feedback gain:

$$\hat{K}_\lambda = -p_\lambda = \alpha\lambda^2 - \sqrt{\alpha^2\lambda^4 + q^2}. \tag{5.77}$$

Note that this is equivalent to the solution previously obtained using our developed closed-loop design methods. Since (A, B) is stabilizable and (A, qI) is detectable, it is known that this controller internally stabilizes the system [16]. We compute an extension of (5.77) to the complex plane, i.e. compute a function $\hat{K}_e$ such that $\hat{K}_e$ recovers $\hat{K}$ when restricted to the imaginary axis. Such an extension is given by the function

$$\hat{K}_e(\sigma) := -\alpha\sigma^2 - \sqrt{q^2 + \alpha^2\sigma^4}, \tag{5.78}$$

which is analytic on the vertical strip of the complex plane defined by the set

$$\{\mathrm{Re}(z) < \sqrt{2}\} \;\subset \mathbb{C}.$$

This region of analyticity is bounded by the branch points of $\sqrt{q^2 + \alpha^2\sigma^4}$ at

$$\sigma = |\sqrt{q/\alpha}| e^{i\theta} \;, \quad \theta = \pi/4, 3\pi/4, 5\pi/4, 7\pi/4.$$

The analyticity region determines the exponential decay rate of the (spatial) convolution kernel of the optimal feedback K for the original LQR problem [16]. This exponential decay rate is interpreted physically as a *localization* of feedback. Note that this exponential decay rate resulted without any constraints imposed on the LQR problem.

5.8 Conclusions & Future Work

In this chapter we provided an operator framework to develop closed-loop parameterizations for general systems over Banach spaces. This general framework allowed us to use SLS like methods to formulate the LQR controller design problem for the diffusion equation over the real line in terms of the resulting closed-loop mappings. This case study demonstrated that a general SLS approach to control of PDEs applies. Although we did not explicitly impose constraints in our example, it is clear that convex constraints on the closed-loops would preserve convexity of this problem. We emphasize though that the constrained problem would only decouple in spatial frequency (as was the case of the unconstrained problem), if the constraints decoupled as well. An open question of interest then is how to design and impose additional constraints on the LQR problem to enforce a more rapid decay rate for controller design for PDEs. Developing an answer to this question is the subject of current and future work.

5.9 Appendix

5.9.1 Proof of Lemma 5.7.1

We use (5.69) to write $\hat{\Phi^u}_\lambda$ in terms of $\hat{\Phi^\psi}_\lambda$ as:

$$\hat{\Phi^u}_\lambda(s) = (s+1)\hat{\Phi^\psi}_\lambda(s) + (\alpha\lambda^2 - 1)\hat{\Phi^\psi}_\lambda(s) - 1. \tag{5.79}$$

If $\hat{\Phi^\psi}_\lambda(s)$ is strictly proper, then $\hat{\Phi^u}_\lambda(s)$ is strictly proper only if $(s+1)\hat{\Phi^\psi}_\lambda(s) - 1$ is strictly proper, i.e. only if $\lim_{s\to\infty}(s+1)\hat{\Phi^\psi}_\lambda(s) = 1$. Then $\hat{\Phi^\psi}_\lambda(s)$ must be of the form: $\hat{\Phi^\psi}_\lambda(s) = \frac{1}{s+1}\left(1 + \hat{\rho}_\lambda(s)\right)$, for some strictly proper $\hat{\rho}_\lambda(s)$. In this case, $\hat{\Phi^\psi}_\lambda(s)$ has finite $\mathcal{H}$ norm iff $\hat{\rho}_\lambda(s)$ has no poles in the closed left half plane. $\hat{\Phi^u}_\lambda(s)$ can be written in terms of the parameter $\hat{\rho}_\lambda(s)$ as $\hat{\Phi^u}_\lambda(s) = \frac{\alpha\lambda^2-1}{s+1} + \frac{s+\alpha\lambda^2}{s+1}\hat{\rho}_\lambda(s)$. Note that this relation shows that if $\hat{\rho}_\lambda(s)$ is stable, then $\hat{\Phi^u}_\lambda(s)$ will have finite $\mathcal{H}$ norm as well. □

5.9.2 Derivation of optimal $\hat{\rho}_\lambda$

(For simplicity of notation, we introduce a new variable $\gamma := \alpha\lambda^2$.) We begin by computing an inner-outer factorization $V_\lambda = V_i V_o$. We obtain V_o as a spectral factor of $V^\sim V$:

$$V^\sim V = V^*(-s)V(s) = \frac{1}{1-s}\begin{bmatrix} q & \gamma - s \end{bmatrix}\frac{1}{1+s}\begin{bmatrix} q \\ \gamma + s \end{bmatrix} = \frac{s - \sqrt{q^2+\gamma^2}}{s-1}\cdot\frac{s+\sqrt{q^2+\gamma^2}}{s+1} =: V_o^\sim V_o \tag{5.80}$$

V_i and $V_i^\sim H_\lambda$ are then computed as

$$\begin{aligned} V_i = V_\lambda V_o^{-1} &= \begin{bmatrix} \frac{q}{s+1} \\ \frac{s+\gamma}{s+1} \end{bmatrix}\frac{s+1}{s+\sqrt{q^2+\gamma^2}} = \frac{1}{s+\sqrt{q^2+\gamma^2}}\begin{bmatrix} q \\ s+\gamma \end{bmatrix}, \\ V_i^\sim H_\lambda &= \frac{1}{\sqrt{q^2+\gamma^2}-s}\begin{bmatrix} q & \gamma - s \end{bmatrix}\frac{1}{s+1}\begin{bmatrix} q \\ \gamma - 1 \end{bmatrix} = \frac{c_1}{s+1} + \frac{c_2}{s-\sqrt{q^2+\gamma^2}}, \end{aligned} \tag{5.81}$$

where $c_1 := \frac{q^2+\gamma^2-1}{1+\sqrt{q^2+\gamma^2}} = \sqrt{q^2+\gamma^2}-1$, and $c_2 := \frac{(\gamma-1)(\sqrt{q^2+\gamma^2}-\gamma)-q^2}{\sqrt{q^2+\gamma^2}+1}$. The projection of $(V_i^{\sim}H)$ onto RH_∞ is $(V_i^{\sim}H)\Big|_{RH_\infty} = \frac{c_1}{s+1}$, so that the optimal solution of (5.71) is

$$\hat{\rho}_\lambda^{\text{opt}}(s) = -V_o^{-1} \cdot \left(V_i^{-1}H\right)\Big|_{RH_\infty} = -\frac{s+1}{s+\sqrt{q^2+\gamma^2}} \cdot \frac{c_1}{s+1} = \frac{1-\sqrt{q^2+\gamma^2}}{s+\sqrt{q^2+\gamma^2}}. \tag{5.82}$$

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