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A convex characterization of distributed control problems in spatially invariant systems with communication constraints

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Abstract

In this paper we consider the problem of distributed controller design in spatially invariant systems for which communication among sites is limited. In particular, the controller is constrained so that information is propagated with a delay that depends on the distance between subsystems—a structure we refer to as "funnel" causality. We show that the problem of optimal design can be cast as a convex problem provided that the plant has a similar funnel-causality structure, and the propagation speeds in the controller are at least as fast as those in the plant. As an example, we consider the case of the wave dynamics with limited propagation speed control.

1. Introduction

We consider spatially distributed systems where all signals are functions of both spatial and temporal variables. The theory of such spatio-temporal systems has been worked out in some detail. We consider only spatially distributed systems with the additional property that the dynamics are spatially invariant. For recent work on this class and some of the background for the present work, we refer the reader to [2–5] and the references therein.

One of the major issues in the design of such distributed controllers is the communications requirements between individual controller sub-systems. One of the applications of this design methodology is to design controllers for large arrays of micro-electro-mechanical system (MEMS), in which there are potentially tens of thousands of actuator/sensor and imbedded control subsystems. For systems of this size and configuration, centralized controllers are not an option. It turns out that optimally designed centralized controllers have an inherent localization property which enables them to be implemented using distributed control elements with limited communication

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requirements [5]. Several researchers have recently been looking at the problem of explicity imposing constraints on communication requirements between controller subsystems. Among these are approaches based on LMIs and convex optimization techniques (see [1,3,4] and the references therein). The same structure of controllers as the plant is imposed and a relaxation is used to obtain stability and performance conditions via LMIs. More recently, an interesting algebraic characterization of plant-controller structures that leads to convex maps has been provided in [7]. The work in [7] nicely generalizes the classes of convex structural problems presented earlier in [9].

In this paper, we consider the case of spatially invariant systems, where the controller is constrained so that information is propagated with a delay that depends on the distance between subsystems—a structure we refer to as "funnel" causality. We show that the problem of optimal design can be cast *exactly* as a convex problem provided that the plant has a similar funnel-causality structure. We also provide explicit constructions of the appropriate coprime factors that lead to a convex model matching formulation for important classes of problems such as the control of systems governed by the wave equation. This work generalizes some of the results in [10] where a special case of funnel causality, termed as cone causality, is considered.

2. Spatio-temporal causality

We begin our exposition by presenting some material on the many possibilities for spatio-temporal causality. Our description of distributed systems will be in terms of their spatio-temporal impulse and frequency responses. A signal $u(x,t)$ is a function of a spatial variable x and a temporal variable t. In what follows, x and t can be either discrete or continuous.

Two signals, u and y are related by a spatially invariant distributed system if we can write

$$y(x,t) = \int \int h(x-\xi, t-\tau) \quad u(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi, \qquad (1)$$

where h is the spatio-temporal impulse response. For uniformity of notation, we use this convolution integral to denote sums as well in the case when $\mathrm{d}\tau$ and $\mathrm{d}\xi$ are discrete measures on the set $\mathbb{Z}$. We will restrict

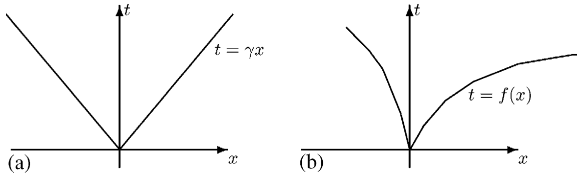

Fig. 1. (a) Cone causality, (b) funnel causality.

attention to the class of temporally causal impulse responses that have the following property:

$$\sup_{t\in[0,T]} \int_{-\infty}^{\infty} |h(x,t)|\,\mathrm{d}x < \infty$$

for any $T > 0$. This can be understood as requiring h to be in L^1 in the spatial coordinate, and in L^∞_{e} (the extended L^∞ space) in the temporal coordinate. This class is large enough to contain most temporally causal (but not necessarily stable) spatio-temporal systems. The above bound allows for composition and inversion of such systems.

The spatio-temporal impulse response $h(x,t)$ can be visualized as a function in the plane (x,t). Temporal causality of h is equivalent to the requirement that $h(x,t)=0$ in the half plane $t \leqslant 0$. Physical systems have temporal causality, but not necessarily spatial causality. As opposed to purely temporal systems, where only one notion of causality is natural, there are many possible notions of causality for spatio-temporal systems. Systems that have a constant finite propagation speed (e.g. the wave equation) are such that h has its support in a "light cone", i.e.

$$h(x,t)=0, \quad \text{for } t < \gamma x,$$

(see Fig. 1a) where $1/\gamma$ is the speed of propagation. This type of causality maybe referred to as "cone causality". We will be considering systems where the impulse response has support in slightly more general domains.

Definition 1. A scalar valued function $f(x)$ is said to be a propagation function if f is non-negative, $f(0)=0$ and such that $\{f(x),\ x \geqslant 0\}$ and $\{f(x),\ x \leqslant 0\}$ are concave functions, respectively.

Definition 2. A system is said to have the property of *funnel causality* if its impulse response is such that

$$h(x,t)=0, \quad \text{for } t<f(x),$$

where $f(x)$ is a propagation function. In other words, if its impulse response is supported in a funnel-shaped region (see Fig. 1b).

Intuitively, any effect in a funnel-causal systems takes at least $f(x)$ time units to travel a distance x. The reason for restricting the propagation function f to have concave segments is that such a class of systems turns out to be closed under convolutions. This latter property will be useful in establishing the convexity of controller design problems later on.

We now state a result on the composition of two funnel-causal systems. If h is a spatio-temporal system, we use the same symbol h to refer to its spatio-temporal impulse response (the function $h(x,t)$), and we use the expression $\operatorname{supp}(h)$ to refer to the region in the (x,t) plane where $h(x,t)$ is supported. If f is a propagation function, we denote by S_f the set

$$S_f := \{(x,t);\ t \geqslant f(x)\}.$$

S_f is the set "above" the curve f in Fig. 1b.

Lemma 1. *Let h_1 and h_2 be two funnel-causal systems such that*

$$\operatorname{supp}(h_1) \subset S_f, \quad \operatorname{supp}(h_2) \subset S_f,$$

where f is some propagation function. Then the composition $h_3 = h_1 * h_2$ is such that

$$\operatorname{supp}(h_3) \subset S_f.$$

Proof. We begin with

$$h_3(x,t) = \int\int h_1(x-\xi, t-\tau)\ h_2(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi.$$

Since $\operatorname{supp}(h_1) \subset S_f$ and $\operatorname{supp}(h_2) \subset S_f$, we have that $h_1(\xi,\tau)=0$ for $\tau < f(\xi)$ and $h_1(x-\xi,t-\tau)=0$ for $t-\tau < f(x-\xi)$ (i.e. $t-f(x-\xi)<\tau$). Thus the limits of integration can be adjusted to

$$h_3(x,t) = \int \int_{f(\xi)}^{t-f(x-\xi)} h_1(x-\xi,t-\tau) \times h_2(\xi,\tau)\,\mathrm{d}\tau\,\mathrm{d}\xi.$$

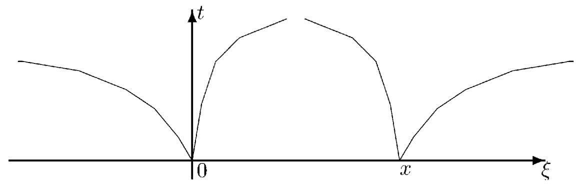

Fig. 2. The function $f(x-\xi)+f(\xi)$.

From this we can in particular conclude that

$$h_3(x,t)=0 \quad \text{if } \forall\xi, \quad t-f(x-\xi) \leqslant f(\xi). \tag{2}$$

To see for which t this condition is valid, note the following implication

$$\forall\xi,\ t \leqslant f(x-\xi)+f(\xi) \Leftarrow \\ t \leqslant \inf_{\xi}\,(f(x-\xi)+f(\xi)). \tag{3}$$

We now claim that the concavity of f implies that

$$\inf_{\xi}(f(x-\xi)+f(\xi)) = f(x). \tag{4}$$

To see this, assume for simplicity that $x>0$, and note that over each of the three intervals $\xi \in (-\infty,0]$, or $[0,x]$ or $[x,\infty)$ the function $\big(f(x-\xi)+f(\xi)\big)$ is the sum of two concave functions (see Fig. 2). Therefore, over each of the three intervals separately, the function $(f(x-\xi)+f(\xi))$ is concave, and its infimum must then be achieved at the boundaries, i.e. at $\xi=0$ or $\xi=x$. In either case, we have Eq. (4), which when combined with (3) and (2) gives

$$h_3(x,t)=0 \quad \text{for } t<f(x),$$

which is the desired conclusion. $\square$

The preceeding lemma characterizes an important property of funnel causal systems. The composition of two such systems is also a funnel-causal system where effects propagate as fast as the fastest of the two systems. To make this precise, let h_1 and h_2 be systems whose support is such that $\operatorname{supp}(h_1) \subset S_{f_1}$ and $\operatorname{supp}(h_2) \subset S_{f_2}$, where $S_{f_1} \subset S_{f_2}$. This means that effects in h_2 propagate faster than in h_1. Lemma 1 then implies that $\operatorname{supp}(h_1 * h_2) \subset S_{f_2}$.

Lemma 1 implies that for a given propagation function, the class of funnel-causal systems is closed under compositions. It is a trivial fact that this class is closed under additions as well. Furthermore, it can also be shown that this class is closed under inversions. The proof of this latter fact is relegated to the appendix. Taking all three properties together, i.e. closure under additions, compositions and inversions, we conclude that the class of funnel causal systems is closed under general linear fractional transformations with coefficients that are themselves funnel causal. This then implies that the Youla et al.-Kucera (YJBK) parameterization (e.g., [8]) can be used to nicely parameterize all funnel-causal stabilizing controllers.

3. Optimal performance and YJBK parametrization

In the design of distributed controllers for spatio-temporal systems it is often desired to impose some decentralized structure on the controller. A fully centralized controller is often impractical in large-scale systems though it has the best performance. Explicit decentralization is a notoriously difficult control problem. Perhaps an indication of the difficulty of this problem is that the set of all achievable closed-loop maps with decentralized control is not in general a convex set. However, we will now consider controllers with prescribed funnel causality which yield convex closed loops for certain plants.

Let a propagation function f be given, and denote by L_f the set of all linear spatially invariant systems with impulse responses that have support in S_f. Consider the standard configuration for disturbance attenuation in Fig. 3 where the plant G and the controller K are spatially and temporally invariant systems. Let g_{22} denote the impulse response of G_{22}, the part of G that maps u to y. A central observation in this paper is that if supp $(g_{22}) \subset S_f$ for some propagation function f, then the problem of designing controllers with support in S_f is convex. The problem of interest is cast as follows.

Optimal performance problem: Consider the standard problem in Fig. 3 with G_{22} such that supp $(g_{22}) \subset S_f$ for some propagation function f. Find the optimal feedback controller with the same funnel-causality constraint as G_{22}, i.e.

$$\inf_{\substack{K \text{ stabilizing} \\ K \in L_f}} \|\mathscr{F}(G; K)\|, \tag{5}$$

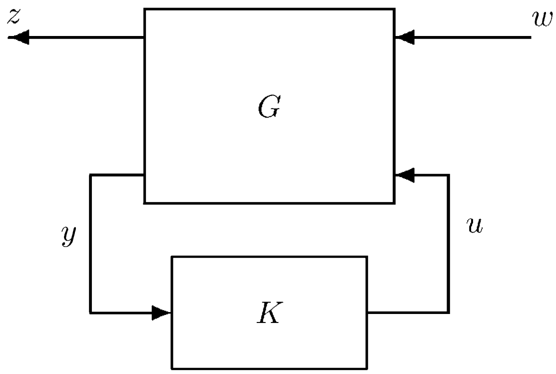

Fig. 3. The standard problem.

where the norm in question is any norm of the closed-loop $\mathscr{F}(G; K)$ defined on spatio-temporal systems.

3.1. Convexity of the set of closed loops

In order to employ the YJBK paramterization for the case of an unstable G_{22} we will assume the existence of a co-prime factorization. We will assume that we can factor $G_{22} = NM^{-1}$, and that there exists X and Y that solve the Bezout identity $XM - YN = I$, where N, M, X, Y are stable spatio-temporal systems. The next results shows that for a funnel-causal system, if a co-prime factorization can be found with funnel-causal factors, then the decentralization constraints on K transform to convex constraints on the Youla parameter Q, which in turn produces a convex set of achievable closed-loop maps. In the next section we explain how we can find Bezout identity factors that are funnel causal for a class of spatio-temporal systems.

Proposition 1. *Let $G_{22} \in L_f$ for some propagation function f. Let $G_{22} = NM^{-1}$ and $XM - YN = I$ with $N, M, X, Y \in L_f$ and stable. Then all stabilizing controllers K such that $K \in L_f$ are given by*

$$K = (Y + MQ)(X + NQ)^{-1}, \tag{6}$$

where Q is a stable system in L_f.

Proof. All stabilizing (possibly without the structure) controllers K are given by $K=(Y+MQ)(X-NQ)^{-1}$, where Q is a stable system without any additional structure. This follows from the standard YJBK argument [8]. Now the class L_f is closed under additions, compositions and inversions. These facts guarantee that $Q \in L_f$ in (6) implies that $K \in L_f$. Conversely, since for any stabilizing controller K, we have $Q := (XK - Y)(M - NK)^{-1}$, then again $K, M, N, X, Y \in L_f$ imply that $Q \in L_f$. $\square$

With the above parametrization, problem (5) becomes

$$\inf_{\substack{Q \text{ stable} \\ Q \in L_f}} \|H - UQV\|, \tag{7}$$

where H, U, V are stable maps that depend only on G. Now since the set of stable $Q \in L_f$ is a linear subspace, and the mapping $Q \mapsto (H - UQV)$ is linear affine, then problem (7) is a convex problem. In particular, it is a minimum distance to a subspace problem. The difficulty of such a problem and whether it is finite or infinite-dimensional will depend on the norm used and the nature of the set L_f (equivalently, the type of propagation function f). A manageable instance of this problem can be found in [10] where the $\mathscr{H}_2$ problem is solved in the case of a relaxed version of cone causality.

4. Some specific factorizations

Co-prime factorizations and Bezout identities can be developed for spatio-temporal systems in a very similar manner to those for finite-dimensional systems. Our interest however is to ensure that the factors and corresponding Bezout identity elements satisfy the same funnel-causality constraints as the plant. Rather than develop the most general procedure for doing co-prime factorization, we concentrate on the special case of plants where decentralized proportional feedbacks can be used to obtain stablizing state feedback and observer gains. This covers a large class of spatio-temporal systems derived from physical partial differential equations.

To begin with, let the input–output distributed system $y = Gu$ be given by a state space realization

$$\begin{aligned} \partial_t \psi &= A\psi + Bu, \\ y &= C\psi, \end{aligned} \tag{8}$$

where ψ, u, and y are spatio-temporal signals, and A, B, C are translation invariant operators. These are operators over L^2 spaces over spatial domains $\mathbb{R}^n$, $\mathbb{Z}^n$ or cross products thereof. B and C are bounded operators, while A is a possibly unbounded operator defined on a dense domain of L^2, and we assume that it generates a C_o (not necessarily stable) semi-group. We refer the reader to [2] for the background and some of the results we later use related to such systems.

We now illustrate how to find co-prime factorizations and solve Bezout identities for such systems. The procedure is very similar to the finite-dimensional case. The Bezout identity is [11]

$$XM - YN = I,$$

where $G = NM^{-1}$, and N, M, X, and Y are stable systems. State space realizations for elements of the Bezout identity are given by

$$\begin{bmatrix} X & -Y \end{bmatrix} = \left[\begin{array}{c|cc} A + LC & -B & L \\ \hline K & I & 0 \end{array}\right], \quad \begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{c|c} A + BK & B \\ \hline K & I \\ C & 0 \end{array}\right] \tag{9}$$

where the spatial operators K and L are chosen such that $A + BK$ and $A + LC$ generate stable evolutions.

The difficulty with obtaining good co-prime factorizations for the problem of funnel causality is that even if the original system is funnel causal, the feedback gains K and L used to form the Bezout identity may destroy this property. We present below a criterion which avoids this problem when simple proportional gains K and L are used.

Proposition 2. *Let a spatio-temporal system be given by the state space model* (8) *such that the impulse responses* $\mathrm{e}^{tA}B$, $C\mathrm{e}^{tA}$ *and* $C\mathrm{e}^{tA}B$ *are funnel causal. If there exists proportional gains K and L* (*i.e. decentralized feedbacks*) *such that* $A + BK$ *and* $A + LC$ *are stable, then all elements of the Bezout identity* (9) *are funnel causal*.

Proof. We consider the right factor in the Bezout identity. Funnel causality of the left factor can be shown

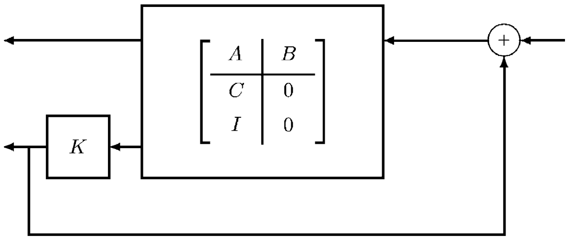

Fig. 4. Feedback decomposition.

similarly. We simply note that the "strictly proper" part (i.e. without the D operator) of this factor can be realized in the feedback diagram shown in Fig. 4. By assumption, the upper part of the feedback diagram is funnel causal, and so is K since it is a decentralized proportional gains. Since funnel-causal systems are closed under compositions, additions and inversions, then any well-posed feedback interconnection of funnel-causal systems is also funnel causal. $\square$

The proof above can be easily generalized to the case when the gains K and L are local spatial operators (e.g. spatial derivatives of any order), as well as when they are any funnel-causal system, but we will not need this generality here. Although it is restrictive to assume that one can find stabilizing decentralized state feedbacks and observer gains, this property seems to hold for a large class of spatio-temporal systems with distributed control. A characteristic example is illustrated in the next subsection. We also note that for vector-valued input and output signals, a non-commutative version of Proposition 1 can be stated. This has the standard form [8], and we do not repeat the formulae here.

Example 1 (*The wave equation*). We illustrate the forgoing ideas using the wave equation. The partial differential equation

$$\partial_t^2 \psi(x,t) = c^2 \partial_x^2 \psi(x,t) + u(x,t), \tag{10}$$

is the standard wave equation with a distributed input. Its transfer function is given by $G(s,k) = (1/s^2 + c^2k^2)$. This system cannot be stabilized by proportional decentralized output feedback alone. A realization of this system has the form (8) as

$$\partial_t \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} 0 & I \\ c^2 \partial_x^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} u,$$

$$\psi = [\, I \quad 0 \,] \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}.$$

Following [2], this system can be analyzed by taking a Fourier transform in the spatial variables. Denoting the spatial Fourier variable by k (the wavenumber), the Fourier representation of the above system is

$$\frac{\mathrm{d}}{\mathrm{d}t} \begin{bmatrix} \psi_1(k,t) \\ \psi_2(k,t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c^2k^2 & 0 \end{bmatrix} \begin{bmatrix} \psi_1(k,t) \\ \psi_2(k,t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(k,t),$$

$$\psi(k,t) = [\, 1 \quad 0 \,] \begin{bmatrix} \psi_1(k,t) \\ \psi_2(k,t) \end{bmatrix},$$

where for simplicity of notation we use the same symbol to denote a signal $\psi(x,t)$ and its spatial Fourier transform $\psi(k,t)$. To see that system (10) has funnel causality, we compute e^{tA}. Note that the 2×2 matrix A can be diagonalized by

$$\begin{bmatrix} 0 & 1 \\ -c^2k^2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \mathrm{i}ck & -\mathrm{i}ck \end{bmatrix} \begin{bmatrix} \mathrm{i}ck & 0 \\ 0 & -\mathrm{i}ck \end{bmatrix} \times \begin{bmatrix} \frac{1}{2} & -\mathrm{i}/2ck \\ \frac{1}{2} & \mathrm{i}/2ck \end{bmatrix}$$

with $c > 0$. This diagonalization then implies that

$$\begin{aligned} \exp\left\{ t \begin{bmatrix} 0 & 1 \\ -c^2k^2 & 0 \end{bmatrix} \right\} &= \begin{bmatrix} 1 & 1 \\ \mathrm{i}ck & -\mathrm{i}ck \end{bmatrix} \begin{bmatrix} \mathrm{e}^{\mathrm{i}kct} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}kct} \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\mathrm{i}/2ck \\ \frac{1}{2} & \mathrm{i}/2ck \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} \mathrm{e}^{\mathrm{i}kct} + \mathrm{e}^{-\mathrm{i}kct} & \frac{1}{\mathrm{i}ck}\left(\mathrm{e}^{\mathrm{i}kct} - \mathrm{e}^{-\mathrm{i}kct}\right) \\ \mathrm{i}ck\left(\mathrm{e}^{\mathrm{i}kct} - \mathrm{e}^{-\mathrm{i}kct}\right) & \mathrm{e}^{\mathrm{i}kct} + \mathrm{e}^{-\mathrm{i}kct} \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}kct} + \mathrm{e}^{-\mathrm{i}kct}\right) & t\,\mathrm{sinc}(kct) \\ -c^2k^2 t\,\mathrm{sinc}(kct) & \frac{1}{2}\left(\mathrm{e}^{\mathrm{i}kct} + \mathrm{e}^{-\mathrm{i}kct}\right) \end{bmatrix} \end{aligned}$$

As is well known, the symbol $\mathrm{e}^{-\mathrm{i}ckt}$ is the Fourier representation of the operator T_{ct} of right translation by distance ct. Multiplication by $t\,\mathrm{sinc}(kct)$ represents convolution with the "rectangular" function

$(1/2c)\mathrm{rec}((1/ct)x)$, where[1]

$$\mathrm{rec}(x) := \begin{cases} 1 & |x| \leqslant 1, \\ 0 & |x| > 1. \end{cases}$$

If we denote by R_{ct} the operation of spatial convolution with $\mathrm{rec}((1/ct)x)$, then we can represent e^{tA} as

$$\mathrm{e}^{tA} = \frac{1}{2} \begin{bmatrix} T_{ct} + T_{-ct} & \frac{1}{c} R_{ct} \\ c\partial_x^2 R_{ct} & T_{ct} + T_{-ct} \end{bmatrix}.$$

Now, ∂_x^2 is a local operator, while T_{ct}, T_{-ct} and R_{ct} are non-local. However, they are all funnel causal with propagation function $f(x) = (1/c)t$ (i.e. they are cone causal). To see this, note that their respective impulse responses are

$$(T_{ct})(x,t) = \delta(x - ct), \quad (T_{-ct})(x,t) = \delta(x + ct),$$
$$(R_{ct})(x,t) = \mathrm{rec}\left(\frac{1}{ct} x\right),$$

all of which are supported in the region $\{(x,t); ct > x\}$.

We have thus established that all elements of e^{tA} are funnel causal. Since B and C are constants, this system satisfies the first set of assumptions of Proposition 2. We now show, how to easily find stabilizing proportional state feedback and observer gains. First, to find a suitable state feedback gain K, note that

$$\begin{aligned} A + BK &= \begin{bmatrix} 0 & 1 \\ -c^2k^2 & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] \\ &= \begin{bmatrix} 0 & 1 \\ -c^2k^2 + k_1 & k_2 \end{bmatrix}. \end{aligned}$$

We set $k_1 = 0$. Then, the eigenvalues of $A + BK$ for each wavenumber k are given by $k_2 \pm \frac{1}{2}\sqrt{k_2^2 - 4c^2k^2}$. Thus for $k_2 < 0$, the spectrum of the operator $A + BK$ is the set $\left[\frac{3}{2}k_2, \frac{1}{2}k_2\right] \cup (k_2 + j\mathbb{R})$, which has negative real part if $k_2 < 0$. Similarly, to find the observer gain, note that

$$A + LC = \begin{bmatrix} l_1 & 1 \\ -c^2k^2 + l_2 & 0 \end{bmatrix}.$$

Setting $l_2 = 0$, we find that the spectrum of $A + LC$ has negative real part if $l_1 < 0$. Choosing $l_1 = k_2 = -1$, we obtain stabilizing gains

$$K = [0 \quad -1], \quad L = \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Now we compute the co-prime factors using formulae (9)

$$\begin{bmatrix} X & -Y \end{bmatrix} = \left[\begin{array}{cc|cc} -1 & 1 & 0 & - \\ -c^2k^2 & 0 & - & 0 \\ \hline 0 & -1 & 1 & 0 \end{array}\right],$$

$$\begin{bmatrix} M \\ N \end{bmatrix} = \left[\begin{array}{cc|c} -0 & 1 & 0 \\ -c^2k^2 & 0 & 1 \\ \hline 0 & -1 & 1 \\ 1 & 0 & 0 \end{array}\right]. \tag{11}$$

We compute the factors to be

$$M = \frac{s^2 + c^2k^2}{s^2 + s + c^2k^2}, \quad X = \frac{s^2 + 2s + c^2k^2 + 1}{s^2 + s + c^2k^2},$$
$$N = \frac{1}{s^2 + s + c^2k^2}, \quad -Y = \frac{-c^2k^2}{s^2 + s + c^2k^2}.$$

The funnel-casuality of all the above factors is guaranteed by Proposition 2.

A closed-loop mapping such as sensitivity can then be written in terms of the Q parameter as

$$(I + GK)^{-1} = XM + NMQ.$$

Example 2 (*Illustration of funnel causality*). The preceding example involved a system whose impulse response is supported in a cone. An example in which the support set is more complex can be constructed from the wave-equation as follows. Consider the addition of two wave-equation-like systems:

$$\partial_t^2 \psi_1(x,t) = c_1^2 \partial_x^2 \psi_1(x,t) + u(x,t), \tag{12}$$

$$\partial_t^2 \psi_2(x,t) = c_2^2 \partial_x^2 \psi_2(x,t) + u(x,t-T), \tag{13}$$

$$\psi(x,t) = \psi_1(x,t) + \psi_1(x,t),$$

where T is a given time delay. The impulse response of this system is simply the sum of the impulse responses of the individual subsystems (12) and (13). The response of (12) is supported in the cone $\{(x,t); c_1 t \geqslant x\}$ while that of (13) is supported in $\{(x,t); c_2(t - T) \geqslant x\}$. Thus, the entire system from u to ψ has an impulse response supported in the set

[1] This is obtained from the Fourier transform pair: $\mathrm{rec}((1/\alpha)x) \leftrightarrow 2\alpha\,\mathrm{sinc}(\alpha k)$.

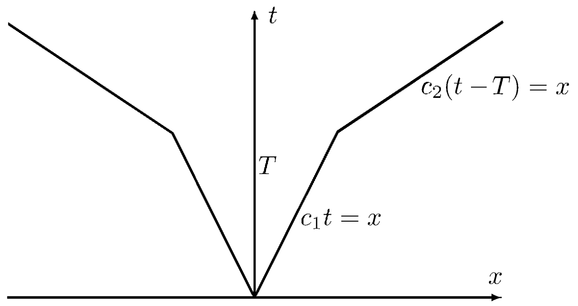

Fig. 5. The impulse response support set of Example 2.

shown in Fig. 5. The figure illustrates the case when $c_1 < c_2$, i.e. when the speed of the ψ_1 system is slower than that of ψ_2.

5. Conclusion

We considered optimal closed-loop design for spatially distributed control where the propagation speeds in the controller are at least as fast as the plant. By characterizing this type of spatio-temporal causality as funnel causality, we have shown these optimal design problems to be convex. For important classes of problems, an explicit construction for deriving the corresponding model matching problem from the original plant data was provided using the YJBK parametrization and state space formulae for the required Bezout identity. This construction guaranteed that the elements of the Bezout identity have the required funnel-causality structure as well. This allows us to handle a large class of spatially distributed unstable systems.

These convex optimal design problems are in general infinite dimensional. Developing efficient procedures for solving or approximating the solutions of these problems is a significant question, and is the subject of current research.

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Appendix. Closure of funnel-causal systems under inversions

In order for the inversion operation to be well defined, we restrict ourselves to the class of temporally causal impulse responses such that for any $T \geqslant 0$,

$$\sup_{t\in[0,T]} \int_{-\infty}^{\infty} |h(x,t)|\,\mathrm{d}x < \infty. \tag{14}$$

This class includes possibly unstable systems.

The class of temporally causal systems is closed under composition. Consequently, if H is temporally causal, then H^n is temporally causal for any n. It is then tempting to define inverses using the Neuman series $(I-H)^{-1} = (I + H + H^2 + H^3 + \cdots)$, and conclude that $(I-H)^{-1}$ must be temporally causal if it exists.

To make sense of the preceeding argument, we must show that the Neuman series converges in some sense. To this end, we can employ a bound similar to that used to show the convergence of successive iteration schemes for Volterra operators [6]. Let h_n denote the impulse response obtained by convolving h with itself n times. The following bound on $h_n(x,t)$ can be established by induction on n:

$$\int |h_n(x,t)|\,\mathrm{d}x \leqslant \frac{t^{(n-1)}}{(n-1)!} \sup_{\tau\in[0,t]} \left(\int |h(x,\tau)|\,\mathrm{d}x\right)^n. \tag{15}$$

The induction argument follows from the following calculation:

$$\begin{aligned}
&\int |h_n(x,t)|\,\mathrm{d}x \\
&\quad = \int \mathrm{d}x \left| \int_0^t \int h_{n-1}(x-\xi, t-\tau) h(\xi,\tau)\,\mathrm{d}\xi\,\mathrm{d}\tau \right| \\
&\quad \leqslant \int_0^t \int \mathrm{d}x \int |h_{n-1}(x-\xi,t-\tau)|\,|h(\xi,\tau)|\,\mathrm{d}\xi\,\mathrm{d}\tau \\
&\quad \leqslant \int_0^t \left(\int |h_{n-1}(x,t-\tau)|\mathrm{d}x\right)\left(\int |h(x,\tau)|\mathrm{d}x\right)\mathrm{d}\tau \\
&\quad \leqslant \int_0^t \frac{t^{n-2}}{(n-2)!} \sup_{\tau\in[0,t]} \left(\int |h(x,\tau)|\,\mathrm{d}x\right)^{n-1} \\
&\qquad \times \sup_{\tau\in[0,t]} \left(\int |h(x,\tau)|\,\mathrm{d}x\right)\mathrm{d}\tau \\
&\quad = \frac{t^{(n-1)}}{(n-1)!} \sup_{\tau\in[0,t]} \left(\int |h(x,\tau)|\,\mathrm{d}x\right)^n,
\end{aligned}$$

where the second inequality follows from the fact that for the L^1 norm $\|f \star g\| \leqslant \|f\| \|g\|$, and the second inequality is the induction step.

Bound (15) implies that the Neuman series converges for any H whose impulse response satisfies (14) to an operator that satisfies (14) over finite time intervals. Note that no bounds on the norm of H were needed (indeed since H is possibly unstable, it may have infinite gain), but causality is necessary. By a simple scaling, the above arguments imply that we can invert any temporally causal system which is of the form $(\alpha I + H)$, where $\alpha \neq 0$ is a scalar. All the cases in which we apply this result are of this form.

The preceeding arguments imply that the Neuman series can be used to characterize inverses of temporally causal systems. Now, if a system H is in addition funnel causal, then Lemma 1 implies that H^n is funnel causal for any n. Therefore, the Neuman series for inversion of H has all terms which are funnel-causal, and consequently, the inverse of H is funnel causal. We note that in this case, since a component of the identity is added to the series, then the “funnel” must include the support of the delta function, i.e. the point $(x, t) = (0, 0)$. An equivalent characterization is that the propagation function must be such that $f(0) = 0$.

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