Frequency Resolution of a Multi Degree of Freedom Resonator

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***Abstract*—This paper outlines the process for estimating measurable parameters in a multi degree of freedom micro resonator. Thermal mechanical noise provides a baseline limit for frequency resolution of micro resonators. We develop the likelihood function for a linear regression model of 2 degree of freedom resonator. Using the Cramer-Rao inequality we determine the minimum possible covariance to estimate parameters, such as the natural frequency of the system.**

I. INTRODUCTION

Micro and nano resonant sensors are utilized in many applications including chemical sensing. For this class of sensors, sensitivity is related to frequency resolution. At this size scale, the ultimate limits of resolution are constrained by thermal-mechanical noise [1]. In this work we have developed a method for calculating the frequency resolution of a multi degree of freedom (MDOF) oscillator, by forming a Fisher Information Matrix and applying the Cramer-Rao inequality which gives the lower bound for the covariance of an estimated parameter [2], such as the natural frequency, i.e. how accurately can you measure the parameter.. The frequency resolution of a single oscillator has been derived using power spectral density [1], and by the Cramer-Rao inequality approach, similar to that presented here [3]. The result presented here is an analytic formula that is implemented to determine the frequency resolution of the 2DOF system given that you can measure the difference between the displacements of the two, or more, coupled oscillators, which we are able to measure using laser Doppler vibrometry. The analytic expression for resolution due to noise aids the design process by quantifying the effect different parameters have on the resolution.

Much attention has been given to studying linear, single degree of freedom oscillators. However, recent projects have focused on MDOF resonators including the single-input-single-output (SISO) mass sensor described in [4]. We are specifically interested in determining the frequency resolution for such a system, shown in Figure 1. There are four micro cantilevers coupled through a larger beam, the shuttle mass. We can calculate the frequency resolution of the shuttle mass alone, and know that the resolution goes down with the coupling of the micro cantilevers. However, to characterize the sensitivity of the device, we must know the frequency resolution of each micro cantilever when we detect through the shuttle mass. With the use of a Polytec MSA-400 laser Doppler vibrometer, we are able to measure the difference between the tip displacement between the shuttle mass and each micro beam, by measuring at the points shown in Figure 1. Since we are able to make this measurement, we study this simpler state, φ=y-x for the model shown in Figure 2, and derived the frequency resolution of the micro beam.

II. SENSOR ANALYSIS

Consider a 2 degree of freedom resonator; this is a simplified model of a SISO sensor. As shown in Figure 2 the large and small oscillators have masses M and m, damping coefficients C and c, stiffnesses K and k, and displacement x and y respectively. The system is driven by a known function $u(t)=U\cos(\omega t)$ and a random force f_n due to thermal mechanical fluctuations in the environment. This gives us the system of equations, which describe the motion.

$$\begin{aligned} &M\ddot{x}+C\dot{x}+Kx+c_1(\dot{x}-\dot{y})+k_1(x-y)=u(t)+(f_N-f_n) \\ &m\ddot{y}+c_1(\dot{y}-\dot{x})+k_1(y-x)=f_n \end{aligned} \quad (1)$$

To investigate the noise forced system, we consider the energy stored in the environment and the mechanical system. When these are at equilibrium, the thermal energy of the environment ($1/2k_BT$ where k_B is Boltzmann's constant and T is the temperature) is equal to the potential ($1/2kx^2$) and kinetic ($1/2mx'^2$) energies of the oscillator. Using the equipartition theorem, the covariance of the random forcing is found to be $r_M=2Ck_BT$ and $r_m=2ck_BT$, for the large and small masses respectively.

Our goal is to use the log likelihood function and the Cramer Rao inequality to obtain the minimum variance that we can measure the natural frequency, or any other parameter, of a two degree of freedom micro resonator [2].

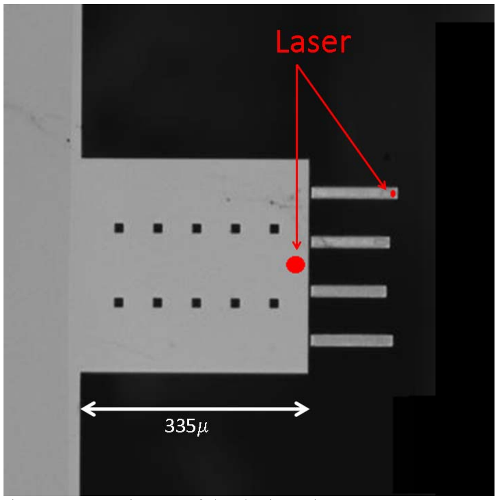

Figure 1: SEM image of the device whose frequency resolution is of interest.

First, we convert the equation of motion into a stochastic differential equation of the form dy=θφdt+dv, where θ is a vector of unknown parameters, φ is the regressor and v is a Wiener process [5] with Edv^2=rdt.

The definition of the Fisher Information Matrix is the derivative of the log likelihood function with respect to parameters θ. First, we insert the regression model into the log likelihood function. Plugging this into the log likelihood function gives us:

$$-\log L = \frac{1}{2r\Delta t}\sum_{k=1}^{N}(\Delta y - \theta^T \varphi \Delta t)^2 = \frac{1}{2r\Delta t}\sum_{k=1}^{N}(e^2(k))^2 \quad (2)$$

Taking the derivative of (2) with respect to θ gives:

$$-\log L_\theta = -\frac{1}{r}\sum_{k=1}^{N} e(k)\varphi \quad , \quad (3)$$

which simplifies to:

$$I = \frac{N}{r}\varphi\varphi^T . \quad (4)$$

Consider again the equations of motion of the oscillators from (1). Since we are able to measure both the difference in displacements and velocities of the oscillators in experiment, we reduce the equations to one state space model:

$$dw = -\frac{c}{m}\varphi_1 dt - \frac{k}{m}\varphi_2 dt + dv \quad (5)$$

where φ_1=y'-x' and φ_2=y-x. From this the Fisher Information Matrix becomes:

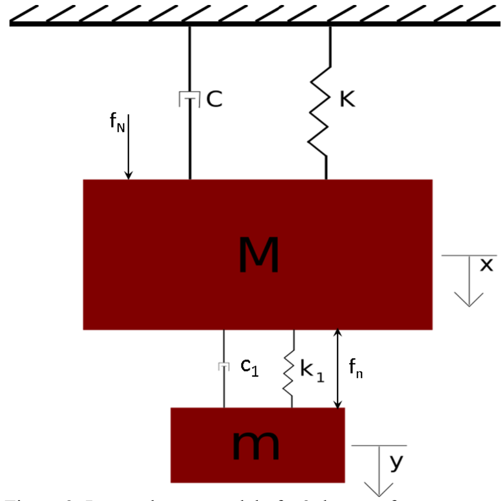

Figure 2. Lumped mass model of a 2 degree of freedom resonator.

$$I = \frac{T_m}{r}\begin{pmatrix} E(\dot{y}-\dot{x})^2 & E(\dot{y}-\dot{x})(y-x) \\ E(\dot{y}-\dot{x})(y-x) & E(y-x)^2 \end{pmatrix}. \quad (6)$$

Now, consider the deterministic and stochastic parts separately, keeping in mind that they will later be combined using superposition.

A. Deterministic Component

To find the deterministic component, first determine the transfer function from the known forcing input to the displacement difference of the oscillators. Here we neglect the noise forces since they have mean value zero. To begin, we take the Laplace transform of the equation of motion. This transforms (1) into

$$\begin{aligned} &\left(Ms^2 + (C+c_1)s + K + k_1\right)X - (c_1 s + k_1)Y = U \\ &-(c_1 s + k_1)X + \left(ms^2 + c_1 s + k_1\right)Y = 0 \end{aligned} . \quad (7)$$

Solving (7) for displacements X and Y gives the transfer functions from the input to the oscillator displacement.

$$\begin{aligned} G_x &= \frac{ms^2 + c_1 s + k_1}{\left(Ms^2 + Cs + K\right)\left(ms^2 + c_1 s + k_1\right) + ms^2(c_1 s + k_1)} \\ G_y &= \frac{c_1 s + k_1}{\left(Ms^2 + Cs + K\right)\left(ms^2 + c_1 s + k_1\right) + ms^2(c_1 s + k_1)} \end{aligned} \quad (8)$$

Subtracting the two, gives the transfer function of interest for this study: the transfer function from the known input u(t) to the displacement difference between the large and small masses.

$$G_{x-y} = \frac{ms^2}{\left(Ms^2 + Cs + K\right)\left(ms^2 + c_1 s + k_1\right) + ms^2\left(c_1 s + k_1\right)} \quad (9)$$

If we excite the system with periodic forcing with frequency ω we find the contribution that the deterministic forcing makes to the Information Matrix is:

$$I_u = \frac{T_m}{2r}\begin{pmatrix} \omega^2 \left|G(i\omega)\right|^2 & 0 \\ 0 & \left|G(i\omega)\right|^2 \end{pmatrix}. \quad (10)$$

Now that we have found the deterministic component, we need to consider the effects of stochastic forcing from thermal mechanical noise.

B. Noise Contributions

For this study, thermal mechanical noise is considered to be the fundamental limit on resolution. To calculate the noise contributions, consider the state model for the system:

$$\frac{d}{dt}\begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -(K+k_1)/M & -(C+c_1)/M & k_1 & c_1 \\ 0 & 0 & 0 & 1 \\ -k_1/m & -c_1/m & -k_1/m & -c_1/m \end{pmatrix}\begin{pmatrix} x \\ \dot{x} \\ y \\ \dot{y} \end{pmatrix} + \begin{pmatrix} 0 \\ N_M/M \\ 0 \\ N_m/m \end{pmatrix} \quad (11)$$

where N_M and N_m are the noise disturbances with incremental covariances $2Ck_BT$ and $2ck_BT$ on the large shuttle & small mass respectively. We now find the steady state covariances by

$$dP/dt = AP + PA^T + R = 0 \quad (12)$$

where A is the dynamics matrix defined in (11) and R is the diagonal covariance matrix

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \frac{2Ck_BT}{M^2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2c_1k_BT}{m^2} \end{pmatrix} \quad (13)$$

The above equations are combined to solve for the entries of the probability matrix **P**, by (12). The entries of **P** are substituted in to the Fisher Information Matrix. This gives us the noise contributions to the Information Matrix

$$I_u = \frac{T_m}{2ck_BT}\begin{pmatrix} p_{22}+p_{44}-2p_{24} & p_{34}-p_{14}-p_{23}+p_{12} \\ p_{34}-p_{14}-p_{23}+p_{12} & p_{11}+p_{33}-2p_{13} \end{pmatrix} \quad (14)$$

This gives all the information needed to calculate the Fisher Information Matrix. Next, we combine the deterministic and noise results using superposition.

III. Cramer Rao Inequality

The noise and deterministic components are combined using superposition to give the full Fisher Information Matrix

$$I = \frac{T_m}{2ck_BT}\begin{pmatrix} \omega^2\left|G(i\omega)\right|^2 + (p_{22}+p_{44}-2p_{24}) & p_{34}-p_{14}-p_{23}+p_{12} \\ p_{34}-p_{14}-p_{23}+p_{12} & \left|G(i\omega)\right|^2 + p_{11}+p_{33}-2p_{13} \end{pmatrix} \quad (15)$$

Taking the inverse of (15) gives us the lower bound to the covariance of the parameters, as defined by the Cramer Rao inequality.

$$\text{cov}(\hat{\theta}) = E(\hat{\theta}-\theta_0)(\hat{\theta}-\theta_0)^T \geq I^{-1} \quad (16)$$

Although the Information Matrix is full, it is simple to numerically calculate the inverse, since it is only a 2x2 matrix.

IV. Conclusion

Using the Fisher Information Matrix and the Cramer Rao inequality, we have successfully developed a design tool to predict the lower bound limit of frequency resolution for a 2 degree of freedom resonator. This tool allows us to better understand the role of geometry and environment on the frequency resolution of the mass sensor. It is widely known that resolution and sensitivity are compromised due to size and coupling. Given the ability to predict these, we can further push the limits for the multi degree of freedom resonator. Current work includes measuring the frequency resolution of the system, as well as expanding this analysis to an N degree of freedom resonator.

This work allows us to directly predict the mass sensitivity for chemical sensors. The mass sensitivity is proportional to the frequency resolution, and related to the natural frequency and the effective mass of the resonator. Since we can calculate all of these, we can now quantitatively predict the sensitivity of the system.

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References

[1] K. Ekinci, Y. Yang, "Ultimate limits to inertial mass sensing based upon nanoelectromechanical systems." J.A.P., vol. 95, no. 5, p. 2682-2689, 2004

[2] K. Shahtalebi, S. Gazor, "On the adaptive linear estimators, using biased Cramer-Rao bound", Signal Processing, vol. 87, nr 6, p. 1288-1300, 2007

[3] Barry E. DeMartini, "Development of Nonlinear and Coupled Microelectromechanical Oscillators for Sensing Applications" (PhD dissertation, UC Santa Barbara, 2008), 103-113.

[4] B. DeMartini, J. Rhoads, et al. "A single input-single output coupled microresonator array for the detection and identification of multiple analytes", J. App. Phys., vol. 93, 2008

[5] B. Oksendale. *Stochastic Differential Equations*. Springer, New York, fifth edition, 2002.

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