The Price of Synchrony: Evaluating the Resistive Losses in Synchronizing Power Networks

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Abstract—In a network of synchronous generators, we investigate resistive power losses due to the power flow fluctuations required to keep the network in a synchronous state. Such fluctuations occur after a transient event or in the face of persistent stochastic disturbances. We term these losses the “price of synchrony”, as they reflect real power flow costs incurred in synchronizing the system. In the case of small fluctuations at each node, we show how the total network’s resistive losses can be quantified using an $H_2$ norm of a system of coupled swing equations subject to distributed disturbances. This norm is shown to be a function of transmission line and generator properties, to scale unboundedly with network size, and to be largely independent of network topology. This conclusion differentiates resistive losses from typical power systems stability notions, which show highly connected networks to be more coherent and thus easier to synchronize. In particular, the price of synchrony is more dependent on a network’s size than its topology. We discuss possible implications of these results to the design of future power grids, which are expected to have highly-distributed generation resources leading to larger networks with the potential for greater transient losses.

I. INTRODUCTION

The electric power system is undergoing rapid changes. The power grid of the future is expected to have higher levels of uncertainty from renewable energy sources [1], changing load patterns [2] and increasingly distributed electricity generation [3]. Many of these changes can affect the stability of the power network. The inherent variability of solar and wind energy, for example, is likely to produce more frequent and higher amplitude disturbances. Such behavior has the potential to affect rotor-angle stability, which is the ability of the power grid to recover synchrony after a disturbance [4]. Synchrony, in this context, refers to the condition when the frequency and phase of all generators within a particular power network are aligned. In other words, when frequencies are equal [5] and phase differences are at an equilibrium state corresponding to balanced power flows throughout the network. Maintaining synchronism, thus depends on the ability to sustain or restore this condition after a disturbance from this nominal operating point.

Synchronization in power systems is typically studied using a so-called network reduced model in which power loads are modeled as equivalent impedances that are absorbed into the ‘transmission line’ for the reduced network [6]–[8]. The resulting system is a set of coupled swing equations that describe the dynamics of a network of generators connected by these lines. This network is then analyzed to determine conditions under which the synchronized state is stable, see e.g. [5], [9]. These problems are related to the well-studied transient stability problem, see e.g. [10], which refers to the ability of a system to return to a stable operating condition after a large angle disturbance.

A recent research trend has been to analyze such problems using tools from systems and control theory. This literature is vast and here we highlight only a subset of this research. In particular, a series of works that draw connections between power grids and coupled Kuramoto oscillators [9], [11]–[13]. The non-uniform Kuramoto oscillator modeling framework provides a first order approximation of the network reduced model. Dörfler and Bullo [12], [13] exploit the properties of this well-studied system to provide network parameter dependent analytical conditions for frequency and phase synchronization in power networks. In a related work [14] these authors also make connections between network reduced models and the structure preserving network models of Bergen and Hill [15]. Similar first order models have been used to investigate the effects of power flow scheduling and increasing power network inter-connectivity (i.e., adding transmission lines) on the rate of convergence [16].

Control design for synchronizing networks of LC-oscillators has also been investigated in [17], [18], where they define synchronization based on voltage differences between connected nodes rather than angle and phase differences. These authors use an $H_2$ system norm as a performance metric for control design, and we emphasize that our problem setting described below uses an $H_2$ of a system with a different set of outputs yielding a different performance metric.

In the present work we formulate a new problem, and study synchrony in power networks in a different context from the more common one. We assume that the network is synchronously stable, i.e., that it is at a stable operating condition and therefore when subjected to a small disturbances it returns to a synchronized state after disturbances. Our main focus is on the control effort required to maintain or return to synchrony. Lack of synchrony leads to non-equilibrium circulating currents [19] passing between generators whose angles are out of nominal phase. These non-equilibrium current fluctuations lead to additional resistive losses over the power lines during the synchronization transient. These losses are essentially the cost of using power flow fluctuations (as opposed to say, a communications infrastructure) as the signaling mechanism to achieve synchronization between generators. It is in this sense that we refer to these losses as a control effort, and as the “price of synchrony”. We point out that other signaling
mechanisms [20] based on information transmission can be used for synchronization, and these would not incur the power losses we investigate. However, in the present paper we do not consider such systems, and focus instead on how the current scheme of using fluctuating power flows as the synchronization signal could scale to larger networks.

The problem just outlined is studied using a reduced network system of synchronous generators subject to disturbances. We show that the total power lost due to non-zero line resistances can be quantified through the $H_2$ norm of an input-output system of coupled swing dynamics with appropriately defined outputs. This $H_2$ norm can be interpreted as the average (per time) power loss when the system is subjected to either persistent stochastic disturbances, or as the total (over all time) power loss due to a transient event. The calculations use a reduction technique using so-called grounded Laplacians [21], [22], by which the neutrally stable network-mean mode (which is unobservable form the performance objective) is removed. Physically, this reduction corresponds to modeling the grounded node as an infinite bus with fixed states.

Our main result proves that the total transient resistive power losses in a network of identical generators depend on a generalized ratio between weighted graph Laplacians defined by the power line resistances and reactances. These losses are shown to increase with network size. In the special case of a network with all edges having identically equal resistance to reactance ratios the losses are entirely independent of the topology and scale directly with the number of nodes. In other words, under these conditions, highly connected and loosely connected networks incur the same resistive power losses in recovering synchrony. Furthermore, these losses grow unboundedly with the number of nodes. Therefore, even though the transient power losses which arise during synchronization are typically a small percentage of the total real power flow, our results show that they may become significant as power networks evolve toward increasingly distributed systems.

Some generalizations of the theory for systems of heterogeneous generators are also provided. In particular, we examine the problem of adding a generator to an existing network. The results indicate that the marginal losses incurred by adding a well-damped, low-inertia generator to the system are small compared those arising through the addition of a poorly-damped, high-inertia one. This generator parameter dependence is particularly relevant in the face of increasingly distributed renewable generation in low voltage grids, which typically have higher resistances (i.e. greater losses). Our results show that merely adding links to the network will not alleviate this effect. However, numerical examples demonstrate that careful selection and placement of a generators can be used to reduce or increase system losses.

The remainder of this paper is organized as follows. Section II introduces the problem formulation and describes connections between the current work and distributed control theory. Section III derives algebraic expressions for the resistive losses and provides the main results. This is followed by a discussion of generalizations and bounds on the $H_2$ norm in Section IV. Section V provides some numerical examples to illustrate the theory. Finally, we summarize the main findings and discuss directions for future work in Section VI.

II. Problem Formulation

Consider a network of $N$ nodes (buses) and a set of edges (network lines) $E$, as depicted in Fig. 1 for a system where $N = 7$. We assume a Kron-reduced network model (see e.g. [10], [23], [24]) where loads are modeled as impedences that are absorbed into the network lines. Thus, at every node $i = 1, \ldots, N$, there is a generator with inertia constant $M_i$, damping coefficient $\beta_i$, voltage magnitude $|V_i|$ and voltage phase angle $\theta_i$. In absence of external control, the dynamics of the $i^{th}$ generator can be described using the following classic machine model [6].

$$M_i \ddot{\theta}_i + \beta_i \dot{\theta}_i = P_{m,i} - P_{e,i} \quad \forall i = 1, 2, \ldots, N,$$ (1)

where $P_{m,i}$ is the mechanical power input from the turbine and $P_{e,i}$ is the electrical power flow out of the $i^{th}$ generator (i.e. the real power injected into the grid), which is given by

$$P_{e,i} = \bar{g}_i |V_i|^2 + \sum_{j \sim i} \sum_{k \sim j} q_{ij} |V_i| |V_j| \cos(\theta_i - \theta_j)$$

$$+ \sum_{j \sim i} b_{ij} |V_i| |V_j| \sin(\theta_i - \theta_j).$$ (2)

Here $j \sim i$ indicates the existence of a line $(E_{ij})$ connecting nodes $i$ and $j$, $g_{ij}$ and $b_{ij}$ are respectively the conductance and susceptance associated with edge $E_{ij}$, and $\bar{g}_i$ is the shunt conductance of bus $i$.

In order to simplify the notation, we define the bus admittance matrix $Y \in \mathbb{C}^{n \times n}$ for the Kron-reduced network as

$$Y_{ij} := \begin{cases} 
\bar{g}_i + \sum_{k \sim i} (g_{ik} - jb_{ik}), & \text{if } i = j, \\
-(g_{ij} - jb_{ij}), & \text{if } i \neq j \text{ and } j \sim i, \\
0, & \text{otherwise}.
\end{cases}$$

$Y$ can be partitioned into a real and an imaginary part such that

$$Y = (L_G + \bar{g}) - jL_B,$$ (3)

where $L_G$ denotes the conductance matrix, $L_B$ denotes the susceptance matrix and $\bar{g} := \text{diag}\{g_i\}$ is the associated diagonal matrix of self conductances. The matrices $L_B$ and $L_G$ are Laplacians of the weighted network graphs respectively.
defined by the susceptances \( b_{ij} \) and conductances \( g_{ij} \) of the lines in the Kron-reduced network.

We now further approximate the power system model (1) and (2) by linearizing the system around a stable operating point \([\theta^*, \omega^*]^{T}\), which WLOG we can transfer to the origin through a change of variables. This linearization allows us to investigate the effects of small disturbances or persistent small amplitude noise within a small neighborhood of this operating point and is well suited to analyzing effects of the small phase angle and frequency changes associated with this system returning to this synchronous state (stable operating point).

The standard linear power flow assumptions include assuming constant voltages, \(|V_i| = 1\) for every bus \(i\), and retaining only the linear terms in (2), which leads to

\[
P_{r,i} \approx \sum_{j \in \mathcal{N}} b_{ij} [\theta_i - \theta_j], \tag{4}
\]

see e.g. [25] for a detailed analysis of the applicability of such assumptions. Substituting (4) into (1) leads to

\[
M_i \ddot{\theta}_i + \beta_i \dot{\theta}_i \approx - \sum_{j \in \mathcal{N}} b_{ij} [\theta_i - \theta_j] + P_{m,i}, \tag{5}
\]

which we can rewrite in state space form as follows.

\[
d \theta \omega \frac{d}{dt} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & I \\ -M^{-1} B & -M^{-1} B \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} w, \tag{6}
\]

where \(M = \text{diag}\{M_i\}\), \(B = \text{diag}\{\beta_i\}\) and we have assumed that \(P_{m,i}\) is a constant that can be lumped into the input \(w\).

**A. System Performance**

As mentioned in the introduction, our concern is not to characterize the stability of the system (6) but rather to evaluate the transient power flows that occur as the system re-synchronizes after a small disturbance from a nominal operating condition or in the face of persistent small amplitude disturbances. We therefore assume the system matrices are such that the dynamics are stable around the equilibrium manifold for which all phases are equal. We now define the system output (measurement) needed to evaluate the real power losses arising from the fluctuating phase angle differences associated with small excursions from the stable operating point.

The real power flow over an edge \(E_{ij}\) is \(P_{ij} = g_{ij} |V_i - V_j|^2\). Since we are regarding \(\theta_i\) as the deviation from the \(i^{th}\) generator's operating point, this power is equivalent to the resistive power loss over an edge during the transient. If we enforce the linear power flow assumptions and retain only the terms that are quadratic in the state variables, then standard trigonometric identities can be used to obtain the quadratic approximation \(P_{ij} \approx g_{ij} [\theta_i - \theta_j]^2\) for these flows. The corresponding sum of instantaneous resistive power losses over all links in the network can then be approximated as

\[
P_{\text{loss}} = \sum_{i \sim j} g_{ij} |\theta_i - \theta_j|^2. \tag{7}
\]

We can now make use of the conductance matrix \(L_G\) to rewrite (7) as the quadratic form \(P_{\text{loss}} = \theta^* L_G \theta\). Since \(L_G\) is a weighted graph Laplacian, which is positive semidefinite, we can define a system output

\[
y = [C_1 \ 0] \begin{bmatrix} \theta \\ \omega \end{bmatrix}, \tag{8}
\]

where \(C_1 := L_G^{-1}\) is the unique positive semidefinite square-root of \(L_G\). It is then easy to see that \(P_{\text{loss}} = y^* y\).

For ease of reference we rewrite the state dynamics (6) and the output equation (8) together as the MIMO LTI system

\[
d \theta \omega \frac{d}{dt} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = - \begin{bmatrix} 0 & I \\ -M^{-1} B & -M^{-1} B \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ M^{-1} \end{bmatrix} w, \tag{9a}
\]

\[
y = L_G^{-1} \begin{bmatrix} \theta \\ \omega \end{bmatrix}. \tag{9b}
\]

This LTI system is a Linear Quadratic approximation of the full nonlinear problem in the sense that the dynamics have been linearized around equilibrium corresponding to the condition where the system power flow in the system is balanced and all generators are operating at a nominal frequency. The instantaneous resistive power losses are quadratically approximated by the (square of the) Euclidean norm of the output signal \(y\). We next describe several interpretations of the \(\mathcal{H}_2\) norm of the system (9) in terms of the total resistive losses (price of synchrony).

**Remark 1:** The system (9a) represents a linearization of the swing dynamics (6) which assumes that line resistances are negligible, while the output (9b) measures the effect of non-zero resistances given the system trajectories arising from (9a).

**B. \(\mathcal{H}_2\) Norm Interpretations for Swing Dynamics**

The LTI system (9) is formulated so that the square of the Euclidean norm of the output \(y(t) + y(t)\) is the instantaneous resistive power loss at time \(t\). The \(\mathcal{H}_2\) norm of this system can be interpreted as the average (per time \(t\)) power loss in a setting with persistent disturbances, or alternatively as the total (over all time) power loss due to a transient event. These interpretations of the \(\mathcal{H}_2\) norm are standard, but we recap them here in the context of the particular physical scenarios for power network setting considered in this paper.

Denote by \(H\) the LTI system (9), and consider the following three scenarios.

i. **Response to a white stochastic input.** When the input \(w\) is a white second order process with unit covariance (i.e. \(E\{w(t)w^*(t)\} = \delta(t - \tau) I\)), the (squared) \(\mathcal{H}_2\) norm of the system is the steady-state total variance of all the output components, i.e.

\[
\|H\|_{\mathcal{H}_2}^2 = \lim_{t \to \infty} E\{y^*(t)y(t)\}. \tag{9}
\]

For the swing dynamics (9) the disturbance vector can be thought of as persistent stochastic forcing at each generator. These disturbances, which are uncorrelated across generators, can be due to uncertainties in local generator conditions, such as changes in local load or supplied mechanical power. The variance of the output is exactly the expectation of the total (over the entire network) instantaneous power loss due to line resistances.
ii. **Response to a random initial condition.** With zero input and an initial condition that is a random variable $x_o$ with correlation $\text{E}\{x_o,x_o^*\} = BB^*$, the $\mathcal{H}_2$ norm is the time integral

$$\|H\|_{\mathcal{H}_2}^2 = \int_0^\infty \text{E}\{y^*(t)y(t)\} \, dt$$

of the resulting response $y$.

The interpretation for (9) is as follows. Since $BB^* = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{M}^{-2} \end{bmatrix}$, which is diagonal, the initial condition corresponds to each generator having a random initial velocity perturbation that is uncorrelated across generators and with zero initial phase perturbation. In this case $\|H\|_{\mathcal{H}_2}^2$ quantifies the total (over all time and the entire network) expected resistive power losses due to the system returning to a synchronized state.

iii. **Sum of responses to impulses at all inputs.** Let $\epsilon_i$ refer to the vector with all components zero except for 1 in the $i^{th}$ component. Consider $N$ experiments where in each, the system is fed an impulse at the $i^{th}$ input channel, i.e. $w_i(t) = \epsilon_i\delta(t)$. Denote the corresponding output by $y_i$. The (squared) $\mathcal{H}_2$ norm is then the total sum of the $L^2$ norms of these outputs, i.e.

$$\|H\|_{\mathcal{H}_2}^2 = \sum_{i=1}^N \int_0^\infty y_i^*(t)y_i(t) \, dt.$$  

A stochastic version of this scenario corresponds to a system where the inputs $w_i$ can occur with equal probability. Under this assumption $\|H\|_{\mathcal{H}_2}^2$ becomes the expected total power loss given these inputs.

The corresponding interpretation for (9) is when each generator is subject to impulse force disturbances (since $w$ enters into the momentum equation of each generator), and $\|H\|_{\mathcal{H}_2}^2$ is the total power loss.

**C. Relations to Network Coherence**

The LTI model (9a) is very similar to the model of vehicular dynamics studied in [26]. The notion of network coherence studied there can be translated in the present context of power networks as quantifying how tightly the phases of all generators drift together. More precisely the following quantity

$$\text{E}\left\{ \left( \theta_i - \frac{1}{N} \sum_{j=1}^N \theta_j \right)^2 \right\},$$

expresses the variance of the deviation of the $i^{th}$ node from the average over all nodes in the network. This quantity is never zero when there are stochastic disturbance inputs, even in a stable power network. Large phase deviation variances reflect a more disordered network while small variances imply a more coherent network.

The $s$ (in network size $N$) of the disorder measure (10) were studied in [26] for regular network structures such as multi-dimensional tori and their variations. The basic trend is the intuitive one that more connected networks tend to be more coherent and vice versa. In that analysis however, the control cost was considered as *per vehicle*, while in the present context, it is the total resistive power loss over the entire network that is of concern. Thus although the two settings have analogous dynamics, the performance objectives differ. We point out that the disorder measure (10) is not the Euclidean norm of the output $y$ defined in (9b). In other words, the amount of phase disorder in a network as measured by (10) is not necessarily related to resistive power losses, and in particular may not scale similarly with network size $N$. While networks with high phase coherence may be desirable for other reasons (such as stability of the nonlinear model), the results to be presented shortly indicate that resistive power losses can be large even in highly coherent networks.

### III. Evaluating Resistive Losses

In this section we derive a formula for the $\mathcal{H}_2$ norm of the system (9) in terms of the system matrices and parameters, then consider the implications for some important special cases. Throughout this section we assume identical generators, i.e., $\mathcal{M} = M I$ and $B = B I$. The first observation is that (9) has the well-known eigenvalue at zero corresponding to the equilibrium condition of all equal states. As is shown in the Appendix, this mode is not observable from the performance output $y$. Thus, if the network is connected, there are no other unstable eigenvalues and the system has a finite $\mathcal{H}_2$ norm. We first go through a system reduction procedure that effectively removes the unobservable mode at 0 and and simplifies the subsequent $\mathcal{H}_2$ norm calculations.

#### A. System Reduction

As previously discussed, $L_G$ and $L_B$ are graph Laplacians. They therefore share the eigenvector $v = 1$, with components all equal to 1, and the associated zero eigenvalue, i.e.,

$$L_B \mathbf{1} = L_G \mathbf{1} = 0.$$  

The zero eigenvalue implies that these matrices are singular and the system (9) is not asymptotically stable. In order to properly define the $\mathcal{H}_2$ norm of (9) we instead regard a reduced system that is asymptotically stable.

Following the approach in [22], we derive the reduced system by first defining a reference state $k \in \{1,\ldots,N\}$. We denote the reduced or grounded Laplacians that arise from deleting the $k^{th}$ rows and columns of $L_G$ and $L_B$ respectively, as $\tilde{L}_G$ and $\tilde{L}_B$. The states of the reduced system $\tilde{\theta}$ are then obtained by discarding the $k^{th}$ elements of each state vector. This leads to a system that is equivalent to one in which $\theta_k = \omega_k \equiv 0$ for some node $k \in \{1,2,\ldots,N\}$ and the physical interpretation is that of connecting the $k^{th}$ node to ground. We call the resulting reduced, or grounded, system $\tilde{H}$

$$\begin{align*}
\frac{d}{dt} \begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix} &= \begin{bmatrix} 0 & I \\ -\tilde{L}_B & -\tilde{L}_G \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ \tilde{w} \end{bmatrix} \\
\tilde{y} &= \begin{bmatrix} \tilde{L}_G^T \\ 0 \end{bmatrix} \begin{bmatrix} \tilde{\theta} \\ \tilde{\omega} \end{bmatrix} =: C\tilde{\phi},
\end{align*}$$

where $\tilde{\phi} = [\tilde{\theta} \quad \tilde{\omega}]^T$. 

A stochastic version of this scenario corresponds to a system where the inputs $w_i$ can occur with equal probability. Under this assumption $\|H\|_{\mathcal{H}_2}^2$ becomes the expected total power loss given these inputs.
Assuming a network where the underlying graph is connected, the grounded Laplacians $L_G$ and $L_B$ are positive definite Hermitian matrices (see e.g. [21]). All of the eigenvalues of system $H$ are thus strictly in the left half plane and the input-output transfer function from $\tilde{w}$ to $\tilde{y}$ has a finite $\mathcal{H}_2$ norm.

B. $\mathcal{H}_2$ Norm Calculation

The squared $\mathcal{H}_2$ norm of the system $H$ is given by

$$||H||^2_{\mathcal{H}_2} = tr(B^*\tilde{X}B),$$

where $\tilde{X}$ is the observability Gramian that can be obtained from the Lyapunov equation $A^*\tilde{X} + \tilde{X}A = -C^*C$. Expanding this expression for the system $H$ in (11) leads to

$$\begin{bmatrix} 0 & -\frac{1}{M} \tilde{L}_B \\ I & -\frac{1}{M^2} I \end{bmatrix} \begin{bmatrix} \tilde{X}_1 \\ \tilde{X}_0 \\ \tilde{X}_2 \end{bmatrix} + \begin{bmatrix} \tilde{X}_0 \\ \tilde{X}_0 \\ \tilde{X}_2 \end{bmatrix} \begin{bmatrix} 0 & 0 & -\frac{1}{M} \tilde{L}_B & -\frac{M}{\beta^2} I \end{bmatrix} = - \begin{bmatrix} \tilde{L}_G & 0 \\ 0 & 0 \end{bmatrix},$$

from which we extract the following two equations

$$X_0 - \frac{\beta}{M} \tilde{X}_2 + \tilde{X}_0 \frac{\beta}{M} = 0 \quad (12a)$$

$$-\frac{1}{M} \tilde{L}_B X_0 + \tilde{X}_0 \frac{1}{M} \tilde{L}_B = -\tilde{L}_G. \quad (12b)$$

Then, using (12a) it is straightforward to compute
$$\frac{d}{dt} tr(\tilde{X}_2) = tr(Re\{\tilde{X}_0\})$$
and (12b) can be rearranged to yield
$$\tilde{L}_B \tilde{X}_0 \tilde{L}_B^{-1} + \tilde{X}_0 = M \tilde{L}_G \tilde{L}_B^{-1},$$
where we make use of the fact that $\tilde{L}_B$ is nonsingular. Combining these and using standard trace relationships leads to the following expression

$$tr(\tilde{X}_2) = \frac{M^2}{2\beta} tr(\tilde{L}_B^{-1} \tilde{L}_G). \quad (13)$$

Finally, noting that $tr(B^*\tilde{X}B) = \frac{1}{M^2} tr(\tilde{X}_2)$, leads to the following Lemma.

**Lemma 3.1:** The squared $\mathcal{H}_2$ norm of the input-output mapping $H$ of the system (11) is given by

$$||\tilde{H}||^2_{\mathcal{H}_2} = \frac{1}{2\beta} tr(\tilde{L}_B^{-1} \tilde{L}_G), \quad (14)$$

where $\tilde{L}_B$ and $\tilde{L}_G$ are the grounded Laplacians obtained using the procedure described above and $\beta$ is each generator’s self damping.

The choice of grounded node $k$ has no influence on the $\mathcal{H}_2$ norm given in (14). We illustrate this point through the following lemmas, which are used derive the main result of Theorem 3.4.

**Lemma 3.2:** Let $H$ denote the input-output mapping (9) with $M = MI$ and $B = \beta I$ and $\tilde{H}$ denote the corresponding reduced system (11). Then, the norm $||H||^2_{\mathcal{H}_2}$ exists and

$$||H||^2_{\mathcal{H}_2} = ||\tilde{H}||^2_{\mathcal{H}_2}.$$ 

**Proof:** See Appendix.

**Lemma 3.3:** Let $\tilde{L}_G$ and $\tilde{L}_B$ be the reduced, or grounded, Laplacians obtained by deleting the $k^{th}$ row and column of $L_G$ and $L_B$ respectively. Then

$$tr(\tilde{L}_B^{-1} \tilde{L}_G) = tr(\tilde{L}_B^{-1} \tilde{L}_G), \quad (15)$$

where $\dagger$ denotes the Moore-Penrose pseudo inverse.

**Proof:** See Appendix.

The result can now be stated in the following theorem, which was also independently derived in [27].

**Theorem 3.4:** Given a system of $N$ generators with equal damping and inertia coefficients $\beta_i = \beta$ and $M_i = M$, $\forall i \in \{1, ..., N\}$ whose input-output response is given by (9). The squared $\mathcal{H}_2$ norm of the system is given by

$$||H||^2_{\mathcal{H}_2} = \frac{1}{2\beta} tr(\tilde{L}_B^{-1} \tilde{L}_G). \quad (16)$$

Thus, the total transient losses of the system are a function of what we term the generalized Laplacian ratio of $L_G$ to $L_B$.

**Proof:** The result follows directly from Lemmas 3.1 - 3.3.

In (9), we assumed that the mechanical input $P_m,i$ to each generator $i$ is lumped into the input $w$. If instead, one chooses to scale the input by the generator’s inertia, i.e., define $w' := P_{m,i}/M$ and $B' := [0\ I]^T$ the squared $\mathcal{H}_2$ norm of the resulting system can be constructed in an analogous manner, as shown in the following Corollary.

**Corollary 3.5:** Consider the modified input-output mapping

$$\frac{d}{dt} \begin{bmatrix} \theta \\ \omega \end{bmatrix} = \begin{bmatrix} 0 & I \\ - \frac{1}{M^2} L_B & - \frac{M}{\beta^2} I \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix} + \begin{bmatrix} 0 \\ I \end{bmatrix} w'. \quad (17)$$

The $\mathcal{H}_2$ norm (squared) of this system is

$$||H'||^2_{\mathcal{H}_2} = \frac{M^2}{2\beta} tr(\tilde{L}_B^{-1} \tilde{L}_G).$$

**Proof:** The result is obtained in an analogous manner to that in Lemma 3.1 and noting that for this modified system $tr(B'^*\tilde{X}B') = tr(\tilde{X}_2)$. The result then follows directly from Lemmas 3.2 and 3.3.

Theorem 3.4 states that resistive losses in a synchronizing network are proportional to what can be thought of as a generalized ratio between the conductance and susceptance matrices. The ratio of power line resistances to reactances in transmission systems is generally small and is often neglected in power flow calculations [25] or in stability analysis of the swing equations. In general, the matrix trace operation in equation (16) implies that resistive losses increase with network size (number of generators). The important special case of equal line ratios presented next shows this relationship clearly. Therefore, resistive losses may become significant in large networks with highly distributed generation even when line resistances are small. In low-voltage micro-grid networks where the resistance to reactance ratios are higher than in
transmission systems this trend would be doubly problematic, as both the network size and ratio would be large.

C. Special Case: Equal Line Ratios

We now consider the assumption that the generalized ratio in (16) is a scalar matrix \( \alpha I \), where

\[
\alpha := \frac{g_{ij}}{b_{ij}} = \frac{r_{ij}}{x_{ij}}
\]

In other words, all lines in the system, \( E_{ij} \in E \) \( \forall i \sim j \), have equal conductance to susceptance, or equivalently resistance to reactance ratios. This assumption implies \( L_G = \alpha L_B \). Thus, by Lemmas 3.1 and 3.2

\[
||H||_2^2 = \frac{1}{2\beta} \text{tr}(\hat{L}_B^{-1} \alpha \hat{L}_B) = \frac{\alpha}{2\beta} (N-1),
\]

which is the result presented in [29]. This result is remarkable in that it says that the loss growth depends only on the network size and is independent of the topology.

Remark 2: Redefining the constant \( \alpha \) (18) as a weighted mean \( \bar{\alpha} \) of the line ratios \( \alpha_{ij} = \frac{g_{ij}}{b_{ij}} \) for all \( E_{ij} \) in the system allows (18) to be generalized to a system with heterogeneous line ratios [27].

Remark 3: A choice of \( \alpha \geq \frac{g_{ii}}{b_{ii}} \) for all edges \( E_{ij} \in E \), makes (18) conservative, and vice versa if \( \alpha \leq \frac{g_{ii}}{b_{ii}} \). Thus, as also noted in [27] using the concept of the weighted mean \( \bar{\alpha} \), the \( H_2 \) norm of the system can be bounded as:

\[
\frac{\alpha_{\text{min}}}{2\beta} (N-1) \leq ||H||_2^2 \leq \frac{\alpha_{\text{max}}}{2\beta} (N-1),
\]

where \( \alpha_{\text{min/(max)}} \) are the smallest (largest) of the line ratios. These bounds also increase unboundedly with the number of generators and are independent of the network topology.

It is worth noting that the topology independence in (18) and the bounds discussed Remarks 2 and 3 are in contrast to measures of power system stability and performance metrics such as network coherence and damping. For example, the topology of the system plays an important role in determining whether a system can synchronize [11], [12], [26], [30] and the network connectivity of a power system is directly related to its rate convergence and damping [16]. One intuitive explanation for the cost of synchrony being independent of network topology is as follows. We expect a highly connected network to have much more phase coherence than a loosely connected network with the same number of nodes. Consequently the power flows per link in a highly connected network are relatively small, but there are many more links than in the loosely connected network. Thus in the aggregate, the total power losses are the same for both networks. A coherent network is however, more stable.

The equal line ratio assumption is not unreasonable for power systems, as the ratio of resistances to reactance ratios of typical transmission links tend to lie within a small interval. In order to quantify this notion we examined four IEEE transmission system benchmark cases and found that a high percentage of the lines fell within a narrow range. For example in the 118 bus system, 90% of the lines had a ratio below 0.34, and 72% lay in the interval 0.20 − 0.30. A recent study [5] suggests that this uniformity in line properties also applies to Kron-reduced networks, by quantifying the homogeneity in node degrees of several reduced actual power networks.

IV. Generalizations and Bounds

In this section, we present further bounds on the expression (16) and discuss their implications for resistive losses. We also address the more general case of non-identical generators.

A. Loss Bounds

As previously mentioned, the term \( \text{tr}(L_B^\dagger L_G) \) in Theorem 3.4 can be interpreted as a generalized ratio between the power network’s conductance matrix \( L_G \) and its susceptance matrix \( L_B \), i.e., the real and imaginary part of the bus admittance matrix without the self conductances, \( Y - \bar{g} \). If we denote the respective eigenvalues of \( L_G \) and \( L_B \) as \( \lambda^G_i \geq ... \geq \lambda^G_2 > 0 \) and \( \lambda^B_i \geq ... \geq \lambda^B_2 > 0 \). The generalized ratio of these two Laplacians can then be lower bounded in terms of their eigenvalues as

\[
\text{tr}(L_B^\dagger L_G) \geq \sum_{i=2}^{N} \frac{\lambda^G_i}{\lambda^B_i}.
\]

(See e.g. [31] for a proof). In the case of identical line ratios, equality holds, and each eigenvalue ratio is equal to \( \alpha \). The unbounded growth of the transient resistive losses with the network size \( N \), which was evident from (18) also applies to (20), since the number of eigenvalues, and thus the sum of their ratios, grows with each added node. We illustrate this growth in the example of Section V-B.

The resistive losses can, as derived in Section III-C, also be lower and upper bounded by (19), which allows for a simple and convenient analysis of the network. These bounds increase unboundedly with \( N \), but become loose if the system is heterogenous in terms of the line resistance to reactance ratios. This may be the case if a combined transmission and distribution network is considered, or if the impedance loads, that are lumped into the lines in the reduced network are very different. In some cases, it is then better to bound the losses in terms of graph-theoretical quantities. This can be done as:

\[
\lambda^G_2 \text{tr}(L_B^\dagger) \leq \text{tr}(L_B^\dagger L_G) \leq \frac{\text{tr}(L_G)}{\lambda^B_2},
\]

where \( \lambda^G_2 \) and \( \lambda^B_2 \) are the algebraic connectivities of the graphs weighted by line conductances and susceptances respectively, see [31] for a proof. It follows that \( \lambda^G_2 \leq \frac{1}{N-1} g_{ii,\text{min}} \) and \( \lambda^B_2 \leq \frac{1}{N-1} b_{ii,\text{min}} \), where \( g_{ii}, b_{ii} \) are the respective self conductances and susceptances of the nodes. Furthermore, the quantity \( \text{tr}(L_B^\dagger) \) is proportional to what we can interpret as the total effective reactance of the network, in analogy with the concept of graph total effective resistance, as recently discussed in e.g. [22] and [32].

By Rayleigh’s monotonicity law (see [33]), the total effective reactance can decrease unboundedly by adding lines and
increasing line susceptances. However, the algebraic connectivity $\lambda_2$ is very small for weakly connected networks and can also be found to decrease with network size. Therefore, while the bounds (21) with $\lambda_2$ in the denominator, are accurate for small, and well-interconnected networks, they become loose for the large, sparsely connected (i.e. not Kron-reduced) networks that most often characterize a power grid.

In a more general context, Theorem 3.4 however applies to all networks with second order consensus dynamics, and the $H_2$ norm can be interpreted as an energy measure [27]. Such dynamics may describe several types of mechanical or biological systems [9], which may be of a different character in terms of edge ratios and conductivities than power systems. The different considerations and bounds discussed here can then be of relevance, especially when the network topology is subject to design, in order to reduce the general energy measure.

B. Systems of Non-Identical Generators

The results derived by considering a grid with identical generators suggest that the losses scale with the network size. In order to put these results in context it is desirable to understand the extent to which these results and the associated analysis can be extended to systems with varying generator properties. In this section we explore these ideas and use the results from previous sections to gain insight for the special case where a non-uniform generator is added to the network.

From Theorem 3.4 we can deduce that

$$1 \frac{1}{2\beta_{\text{max}}} \text{tr} \left( L_B^* L_G \right) \leq \| H_2 \|_2^2 \leq \frac{1}{2\beta_{\text{min}}} \text{tr} \left( L_B^* L_G \right),$$

where $\beta_{\text{min}} = \min_{i \in \{1,...,N\}} \beta_i$ and $\beta_{\text{max}} = \max_{i \in \{1,...,N\}} \beta_i$. The losses are thus bounded by the properties of the and most strongly and lightly damped generators respectively. Some interesting questions that arise from this observation are: (1) How does adding a generator to an existing network effects the total resistive losses? and (2) What are the important parameters in determining this incremental cost? The next result addresses one such scenario.

***Lemma 4.1***: Consider a network of $N$ generators with transient resistive losses given by $\| \hat{H}_0 \|_2^2$. If one connects an additional generator with damping $\beta_{N+1}$ and inertia $M_{N+1}$ to any node $k \in \{1,...,N\}$ in the existing network by a single link with line ratio of $\alpha_{k,N+1} = \frac{r_{k,N+1}}{x_{k,N+1}}$. Then the new network’s losses are given by

$$\| \hat{H}_1 \|_2^2 = \| \hat{H}_0 \|_2^2 + \frac{1}{2\beta_{N+1}} \alpha_{k,N+1}.$$  

If the dynamics are as per (17) the additive term is instead

$$\frac{M_{N+1}}{2\beta_{N+1}} \alpha_{k,N+1}.$$ 

***Proof:*** See Appendix.

This result can be interpreted as follows: the additional losses incurred through the connecting a “light” (low inertia) or well damped generator are smaller than those incurred due to adding a “heavy” (high inertia) or poorly damped generator. In the face of increased penetration of renewable generation, this result implies that while the synchronization losses do scale with the network size, the impact of typically low inertia renewable generators is relatively low, compared to that of conventional generators. This situation is explored in the example of Section V-C.

Although this is the only analytically result for non-homogeneous generator parameters, the numerical example in Section V-D provides further insight into the “price of synchrony” in networks of non-uniform generators. That example shows that although the scaling relationships and topology independence results for the special cases discussed herein hold for small parameter variations, judicious sizing and placement of new generators can in fact reduce system losses.

***Remark 4***: While results in this paper are only valid for Kron-reduced networks of synchronous generators modeled by second-order swing equations, ongoing work [34] using an extended model of a structure-preserving network with load dynamics and asynchronous generators indicates that the main results presented in this paper extend to these more general systems. Those results imply that the Kron-reduction of the network employed here does not account for the topological independence or network scaling observed in this work.

V. NUMERICAL EXAMPLES

The results derived and discussed in the previous sections indicate that the resistive losses in a network of generators depend on the number of generators in the system, the system’s resistance to reactance ratios and the generator properties. In this section, we provide some numerical examples to illustrate these results.

A. Line Ratio Variance

Fig. 2 shows the resistive losses according to Theorem 3.4 for a hypothetical set of identical generators ($\beta = 1$) with the topologies of the IEEE 14 bus and 30 bus benchmark systems [35] and line ratios $\alpha_{ij} = \frac{r_{ij}}{x_{ij}} = \frac{b_{ij}}{y_{ij}}$ for $E_{ij} \in \mathcal{E}$ drawn from uniform distributions on the intervals $0.4, 0.4 \pm 0.025, 0.4 \pm 0.05, ..., 0.4 \pm 0.2$. Here, the horizontal axis indicates the resulting standard deviation of the line ratios. For all of these cases we take the values for $x_{ij}$ from the
benchmark systems and set $r_{ij} = \alpha_{ij}x_{ij}$. The bars in Fig. 2 represent the upper and lower bounds of the inequality (19).

We note that while increased standard deviation of the line ratios leads to a looser bound in (19), the resistive losses of the system themselves vary very little when the average line ratio remains constant. They are instead highly dependent on the network size (here 14 or 30 nodes), which is consistent with the relationship (18) derived in [29]. These small changes in the value of the norm as the variance of the parameters is increased can also be understood by considering Theorem 3.4 with the conductance matrix decomposed as $L_G = \alpha L_B + \bar{L}_G$. Then (16) can be rewritten as

$$||H||^2_{\mathcal{H}_2} = \frac{\alpha}{2\beta}(N - 1) + \frac{1}{2\beta} \text{tr} \left( L_B^{\dagger} \bar{L}_G \right),$$

which shows that the size of the deviation from the equal line ratios depends the size of $\text{tr} \left( L_B^{\dagger} \bar{L}_G \right)$. In Fig. 2, this quantity is the small deviations of the points from the horizontal lines representing equal line ratios. In general, the entries of $L_G$ take on both positive and negative values, and when the line ratios are close to $\alpha$ their absolute values are small and these values increase as their deviation from $\alpha$ increases. For a meaningful choice of $\alpha$, such as the average value the term $\text{tr} \left( L_B^{\dagger} \bar{L}_G \right)$ will thus be small.

### B. Increased Network Size

According to our results, the resistive losses in a network of synchronizing generators are largely independent of the network topology, but instead depend on the ratio between susceptances and reactances of the lines of the Kron-reduced network. Further they increase unboundedly with the network size. In this example, we will consider the $\mathcal{H}_2$ norm of an increasingly large radial network to that of a growing network described by a complete graph, in which every node is connected to every other node.

For simplicity, we regard the case where $\beta = 1$, and assign random line parameters to the increasingly large networks, letting each line’s reactance $x_{ij}$ and line ratio $\alpha_{ij}$ both be drawn from a normal distribution with mean 0.2 and standard deviation 0.1 (replacing any negative values with the mean). As shown in the previous example, one can then expect the norm for each network to strongly depend on the mean ratio $\bar{\alpha} = 0.2$ and the number of nodes $N$.

Fig. 3 shows how the norm increases for the radial and the complete graphs as the network size increases from a 5 node to a 50 node system, with the bounds (19), (20) indicated on both panels and (21) shown only for the complete graphs. The eigenvalue ratio bound (20) provides the tightest bound for the complete graphs as the network size increases from a 5 node to a 50 node system, with the bounds (19), (20) indicated on both panels and (21) shown only for the complete graphs. The eigenvalue ratio bound (20) provides the tightest bound and grows with $N$ in the same fashion as the norm. We also note that the network-parameter dependent bounds (21) are more accurate than the line ratio bounds (19) for the complete graph.

### C. Generator Parameter Dependence of Incremental Losses

To characterize a case with non-uniform generators, we will now study the situation in Lemma 4.1 and simulate the 7 bus network depicted in Fig. 1. We let all lines $E_{ij} \in E$ have the impedances $z_{ij} = z_0 = 0.04 + j0.2$, node 1 be the grounded node, and let all generators $i = 2, ..., N = 7$ have the parameters [36]: $M_i = \frac{20}{2\pi\beta_0}$ and $\beta_i = \frac{10}{2\pi\beta_0} = \beta_0$ with a frequency $f = 60$ Hz. Let this original system be denoted by $H_0$.

If three additional generators are connected at node 1, each with a different $\beta_{N+1} = \beta_8 \in \{0.1\beta_0, \beta_0, 10\beta_0\}$, the connecting line has the impedance $z_{1,8} = z_0$. Fig. 4 shows the system trajectories of the three resulting reduced systems $H_1$, when they are subjected to a random initial angular velocity disturbance, corresponding to the $\mathcal{H}_2$ norm interpretation (ii) in Section II-B.

The expected power losses during the transient response for these respective systems are given by Lemma 4.1 as $||H_1||^2_{\mathcal{H}_2} = ||H_0||^2_{\mathcal{H}_2} + \frac{\pi}{2\beta_0} = 60.3$, 26.4 and 23.0 respectively. For the particular example in Fig. 4, the losses are respectively 110, 32.2 and 23.7. The weakly damped generator will experience strong oscillations and incur large losses before
it stabilizes at the same states as the grounded node. The highly
amped generator, on the other hand, incurs less oscillations
and losses than in the case where a generator identical to the
ones in the system is connected.

D. Networks With Non-Uniform Generation

Our final example relaxes the assumption of equal generator
parameters. Consider the network depicted in Fig. 1 with equal
line impedances $r_{ij} + jx_{ij} = 0.1 + j0.6$. In this network,
nodes 1 and 7 have the smallest degree, which we denote
$D_1$. Node 4 then has degree $4D_1$, nodes 2 and 6 have degree
$3D_1$ and nodes 3 and 5 both have degree $2D_1$. We set the
generator parameters to $M = \frac{20}{2\pi}$ for all nodes and $\beta \in \frac{1}{2\pi}\{2, 8, 14, 20\}$ for each of the 4 types of nodes, i.e.
nodes having degrees $D_1, 2D_1, 3D_1$ or $4D_1$.

We study this system under two conditions (a) the strongly
damped generators are placed at highly interconnected nodes
(i.e., matched dampings and degrees) and (b) strongly damped
generators are placed at the least connected nodes (i.e., mis-
matched dampings and degrees). We define $H_{match}$ as the
system corresponding to a network where the node degrees
have been matched to the size of the damping coefficients $\beta$
(i.e. $\beta_i = 2$ at nodes degree $D_1$, $\beta_i = 8$ at nodes with degree
$2D_1$, $\beta_i = 14$ at nodes with degree $3D_1$ and $\beta_i = 20$ at nodes
with degree $4D_1$) as in condition (a), and define the system
$H_{mismatch}$ corresponding to a network where the degrees and
damping are mismatched as in condition (b).

These systems are subjected to a random initial angular
velocity disturbance in the manner described in Section V-C.
This leads to $\|H_{match}\|_{H_2}^2 = 18.9$ and $\|H_{mismatch}\|_{H_2}^2 =
20.7$, which indicates that there are lower losses for the system
corresponding to case (a) where the dampings are matched to
the nodal degrees.

Fig. 5 shows the state trajectories of the two systems for
a particular input sequence. For this particular example, the
transient behaviour of the system $H_{mismatch}$ is clearly less
"coherent" than that of $H_{match}$, and because the connectiv-
gy of the graph underlying these networks is identical the
additional oscillations in phase angle will lead to increased
transient losses. This observation is verified when we compute
the respective losses for the trajectories shown. These are 13.2
for the matched case in Fig. 5a and 27.9 for the mismatched
case in Fig. 5b.

These results and similar case studies have led us to con-
clude that for systems with non-uniform generator parameters,
judicious network design that places well-damped generators
at highly interconnected nodes can reduce transient power
losses. An intuitive explanation to this is that a well-damped
generator is able to exert a larger effect on the entire network
if it is well-interconnected than if it is remotely located.

VI. CONCLUSIONS

We quantified the resistive line losses that occur due to
the power flows required to maintain phase synchrony in a
power network in the presence of persistent disturbances or
transient responses. These losses are the cost of using power
flow through transmission lines as the signaling mechanism
for phase synchronization control, which motivates the term
"price of synchrony". In the special case of identical gen-
erators, we derived a formula for the total losses expressed
as a generalized ratio of the weighted graph Laplacians of
the conductance and susceptance matrices. We showed that
this quantity generally scales unboundedly with the number
of nodes (generators) in the system. For the special case
where all of the transmission lines have equal conductance
to susceptance ratios, we showed that the total resistive losses are
independent of network topology, and directly proportional to

\begin{align*}
\text{(a) } & \beta_{N+1} = 10\beta_0 \\
\text{(b) } & \beta_{N+1} = \beta_0 \\
\text{(c) } & \beta_{N+1} = 0.1\beta_0
\end{align*}

Fig. 4: Simulation of the grounded 7 bus network of Fig. 1 with identical generators in the main network and one additional
generator of (a) 10 times, (b) once, and (c) a tenth of the damping of the other generators connected to the grounded node number
1. The system is subject to random velocity and zero phase initial conditions, so that the expected power losses correspond to
the $H_2$ norm. The losses are the largest in system (c), where the lightly damped generator maintains its oscillation for a very
long time, as predicted by Lemma 4.1.
the number of nodes in the network. This topological independence implies that while a highly connected network may have better phase coherence and transient stability properties than a loosely connected one, the two types are equivalent in terms of the transient power losses required to maintain that phase coherence. While this conclusion may at first seem surprising, it becomes fairly intuitive when one considers the following contrast between a highly versus sparsely connected networks. A highly connected, and therefore highly phase-coherent network has much smaller phase fluctuations than a loosely connected one. Therefore, while the “per-link” resistive losses are smaller in former, it has many more links than the latter, and thus the total losses summed over all links are the same for both networks. While, ongoing continued work [34] shows the scaling relationships are retained in network-preserving model with load and asynchronous generator dynamics, an important future research question is to determine the extent to which these results can be generalized to general power networks.

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APPENDIX

Proof of Lemma 3.2

Consider the following state transformation of the system (9),

\[
\begin{bmatrix}
\theta' \\
\omega'
\end{bmatrix} = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix} \begin{bmatrix} \theta \\ \omega \end{bmatrix},
\]

where \( U \) is the unitary matrix which diagonalizes \( L_B \), i.e.,

\[
U^* L_B U = \Lambda_B = \text{diag} \{ \lambda_0^B, \ldots, \lambda_N^B \},
\]

where \( \lambda_0^B \leq \lambda_2^B \leq \ldots \leq \lambda_N^B \) are the eigenvalues of \( L_B \). We have assumed, without loss of generality, that \( U = \left[ \frac{1}{\sqrt{N}} \ 1 \\ u_2 \\ \ldots \\ u_N \right] \), where \( u_i, i = 2, \ldots, N \) are the eigenvectors corresponding to the aforementioned eigenvalues.

Since the \( H_2 \) norm is unitarily invariant, we can also define \( w' = U^* w \) and \( y' = U^* y \) to obtain the system

\[
\frac{d}{dt} \begin{bmatrix} \theta' \\ \omega' \end{bmatrix} = \begin{bmatrix} -\frac{1}{M} \Lambda_B & I \\ -I & \frac{1}{M} \end{bmatrix} \begin{bmatrix} \theta' \\ \omega' \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix} w'.
\]

(22)

Now, observe that

\[
U^* L_G U = \begin{bmatrix} 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \hat{L}_G \end{bmatrix},
\]

(23)
which implies that the first rows and columns of both $U^*L_GU$ and $\Lambda_N^I$ are zero. We thus have that the states $\theta_1' = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \theta_i$ and $\omega_1' = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} \omega_i$ satisfy the dynamics

\begin{align}
\dot{\theta}_1' &= \omega_1', \quad (24a) \\
\dot{\omega}_1' &= -\frac{1}{M} \omega_1' + \frac{1}{M} w_1' \quad (24b) \\
y_1' &= 0. \quad (24c)
\end{align}

Equation (24) reveals that the associated mode, which we denote $H_1'$ and that corresponds to the single zero eigenvalue of $L_B$, is unobservable from the output. The remaining eigenvalues of the system (22) lie strictly in the left half of the complex plane because $L_B$ is positive semidefinite. It follows that the input-output transfer function from $w'$ to $y'$ is stable and has finite $\mathcal{H}_2$ norm.

By the equivalence of this system and $H$, we have thus established the existence of the $\mathcal{H}_2$ norm for the system $H$.

We can now partition the system into the subsystems $H_1'$ and $\hat{H}$. We take $\hat{L}_G$ as the Hermitian positive definite submatrix in (23) and define $\Lambda_B = \text{diag}\{\lambda_B^1, \lambda_B^2, ..., \lambda_B^N\}$ and write the input-output mapping $\hat{H}$ as:

\begin{align}
\frac{d}{dt} \begin{bmatrix} \hat{\theta} \\ \hat{\omega} \end{bmatrix} &= \begin{bmatrix} 0 & 0 \\ -\frac{1}{M} \hat{\Lambda}_B - \frac{1}{M} I \end{bmatrix} \begin{bmatrix} \hat{\theta} \\ \hat{\omega} \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} I \end{bmatrix} \tilde{w} \\
\dot{\tilde{y}} &= \begin{bmatrix} \hat{L}_G^T \\ \hat{L}_G^T \end{bmatrix} \begin{bmatrix} \hat{\theta} \\ \hat{\omega} \end{bmatrix} \quad (25)
\end{align}

or $\frac{d}{dt} \begin{bmatrix} \hat{\theta} \\ \hat{\omega} \end{bmatrix} = A\hat{\theta} + B\hat{\omega}; \quad \dot{\tilde{y}} = C\phi.$

Note that the systems $H_1'$ and $\hat{H}$ are completely decoupled and we therefore have that $\|H\|_{\mathcal{H}_2}^2 = \|H_1'\|_{\mathcal{H}_2}^2 + \|\hat{H}\|_{\mathcal{H}_2}^2 = \|\hat{H}\|_{\mathcal{H}_2}^2$.

The $\mathcal{H}_2$ norm can then be calculated in perfect analogy to the derivations in Section III-B and we obtain that

$$\|H\|_{\mathcal{H}_2}^2 = \frac{1}{2\beta} \text{tr}(\hat{\Lambda}_B^{-1}\hat{L}_G^T).$$

Now, we show that the result of Lemma 3.1 can be written in terms of the state transformed matrices $\Lambda_B$ and $\hat{L}_G$. Define the $N \times (N-1)$ and the $(N-1) \times N$ matrices $R$ and $P$ by:

$$R = \begin{bmatrix} 0 & \cdots & 0 \\ I_{N-1} & \cdots & 0 \end{bmatrix}, \quad P = \begin{bmatrix} I_{k-1} \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

where $k$ is the index of the grounded node and $-1$ is the $(N-1) \times 1$ vector with all entries equal to $-1$. By this design, $\Lambda_B = R^*\Lambda_B R$, $\hat{L}_G = R^*U^*L_G U/R$ and $L_B/G = P^*L_B/G P$. Further, to simplify notation, we define the $(N-1) \times (N-1)$ non-singular matrix $V = PU/R$. Then we can write

$$\text{tr}(\hat{L}_G^{-1}) = \text{tr}(V^{-1}\hat{L}_G^{-1}(V^*)^{-1}V^*\hat{L}_G),$$

since $V^{-1} = (V^*)^{-1}V^* = I$. By the cyclic properties of the trace:

$$\text{tr}(V^{-1}\hat{L}_G^{-1}(V^*)^{-1}V^*\hat{L}_G) = \text{tr}(V^{-1}\hat{L}_G^{-1}(V^*)^{-1}V^*\hat{L}_G V) = \text{tr}((V^*\hat{L}_G V)^{-1}V^*\hat{L}_G V).$$

But $V^*\hat{L}_G V = R^*U^*P^*\hat{L}_G P U R = \hat{\Lambda}_B$ and $V^*\hat{L}_G V = R^*U^*P^*L_G P U R = \hat{L}_G$. Hence,

$$\text{tr}(\hat{L}_G^{-1}) = \text{tr}(\hat{\Lambda}_B^{-1}).$$

In conclusion,

$$\|H\|_{\mathcal{H}_2}^2 = \frac{1}{2\beta} \text{tr}(\hat{\Lambda}_B^{-1}\hat{L}_G^T) = \frac{1}{2\beta} \text{tr}(\hat{L}_B^{-1}\hat{L}_G) = \|\hat{H}\|_{\mathcal{H}_2}^2,$$

which proves the Lemma.

**Proof of Lemma 3.3**

By the proof of Lemma 3.2, we have that $\text{tr}(\hat{L}_B^{-1}\hat{L}_G) = \text{tr}(\hat{\Lambda}_B^{-1}\hat{L}_G)$.

Now,

$$\text{tr}(\hat{\Lambda}_B^{-1}\hat{L}_G) = \text{tr}\left(\begin{bmatrix} 0 \\ \hat{\Lambda}_B^{-1}\hat{L}_G \end{bmatrix}\right) = \text{tr}\left(\begin{bmatrix} 0 \\ \hat{\Lambda}_B^{-1}\hat{L}_G \end{bmatrix} U^*L_G U\right).$$

By definition, see e.g. [37], $U^*L_G U = \text{diag}\{\frac{1}{N}, \frac{1}{N}, ..., \frac{1}{N}\}$, which makes the above equivalent to: $\text{tr}(U^*L_B^1 U U^*L_G U) = \text{tr}(U^*L_B^1 U L_G U)$. But since the trace is unitarily invariant, it follows that

$$\text{tr}(\hat{\Lambda}_B^{-1}\hat{L}_G) = \text{tr}(L_B^1 L_G),$$

which concludes the proof.

**Proof of Lemma 4.1**

Without loss of generality, choose the node the new generator is connected as the ground node, and denote it by $N$. Let $\mathcal{M} := \text{diag}\{M_1, ..., M_N\}$, $\mathcal{B} := \text{diag}\{\beta_1, ..., \beta_N\}$ and denote the new node $(N+1)^{th}$ as $NI$ for notational compactness. The reduced system $\hat{H}_1$ can then be written as

$$\frac{d}{dt} \begin{bmatrix} \theta_{NI} \\ \omega_{NI} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ \mathcal{M}^{-1} \hat{L}_B \\ \mathcal{M}^{-1} \mathcal{B} \\ \mathcal{M}^{-1} \omega_{NI} \end{bmatrix} \begin{bmatrix} \tilde{w} \\ \omega_{NI} \end{bmatrix}.$$  \quad (27)

Let the input-output mapping $\hat{H}_1$ be the SISO subsystem of (27):

$$\frac{d}{dt} \begin{bmatrix} \theta_{NI} \\ \omega_{NI} \end{bmatrix} = \begin{bmatrix} \theta_{NI} \\ \omega_{NI} \end{bmatrix} + \begin{bmatrix} \frac{1}{\mathcal{M}_{NI}} \\ \mathcal{M}_{NI} \end{bmatrix} \begin{bmatrix} \theta_{NI} \\ \omega_{NI} \end{bmatrix}.$$  \quad (27)

From (27), it is clear that the systems $\hat{H}_0$ and $H_{NI}$ are entirely decoupled and since we can write $\hat{H}_1 = \text{diag}\{\hat{H}_0, \hat{H}_1\}$,

$$\|\hat{H}_1\|_{\mathcal{H}_2} = \|\hat{H}_0\|_{\mathcal{H}_2} + \|H_{NI}\|_{\mathcal{H}_2}^2.$$

Now, the $\mathcal{H}_2$ norm of $H_{NI}$ can be calculated in scalar analogy to the derivation in Section III-B to yield

$$\|H_{NI}\|_{\mathcal{H}_2}^2 = \frac{1}{2\beta_{NI}} \frac{\mathcal{M}_{NI}}{\mathcal{B}_{NI}} \|\hat{N}_{NI}\|_{\mathcal{H}_2}^2,$$

which concludes the proof.